Vectors and Linear Transformations

Ling 282/482: Deep Learning for Computational Linguistics
C.M. Downey
Fall 2025

Linear Algebra Objects

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- **Scalars**
  - Single numbers
  - What you're used to elsewhere in math
  - examples: 0, 1, 3.14, π, 7/22

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$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

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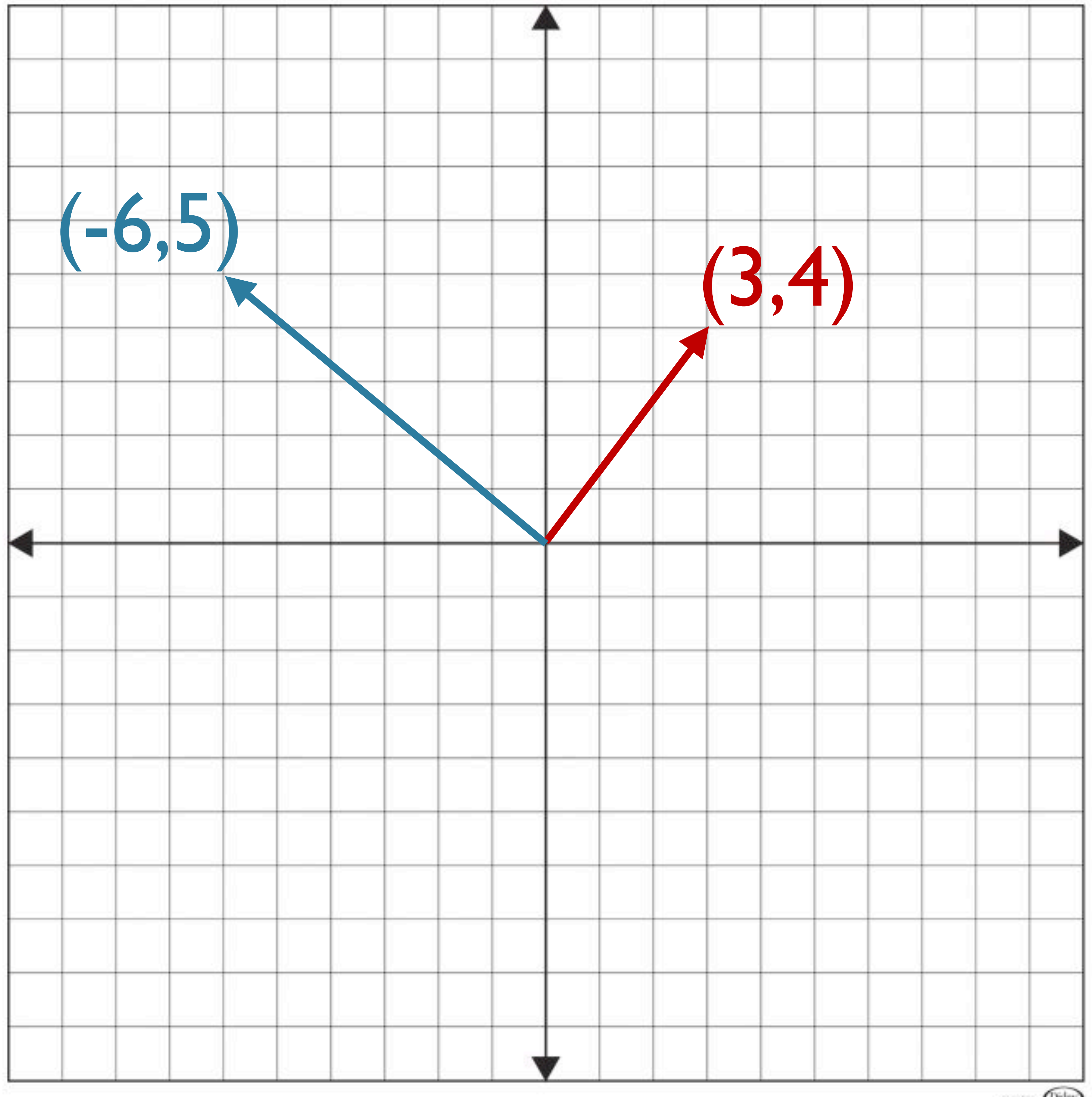

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$$c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$$

(c is a scalar)

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- The dot product returns a **single value!!** (not a vector)
- Intuition: the **strength** with which the two vectors **go in the same direction**
    - (Not super important to remember this right now)
    - Important fact: **perpendicular vectors** have a dot product of **zero!**

Vector Spans and Spaces

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- Note: $a = \mathbf{0}$ is used to indicate a vector of zeros

Vector (In)dependence Examples

- What constants solve this equation?

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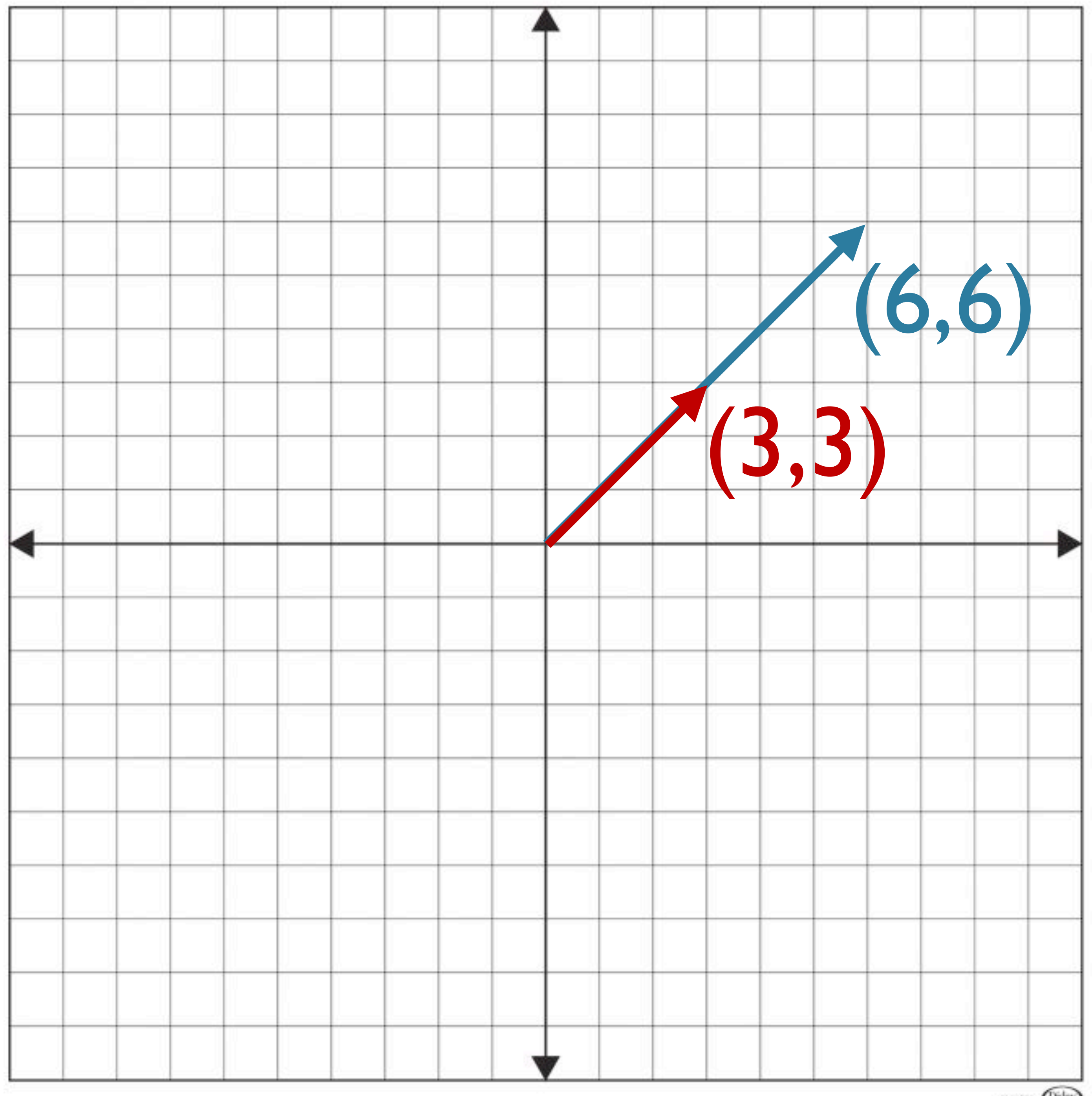

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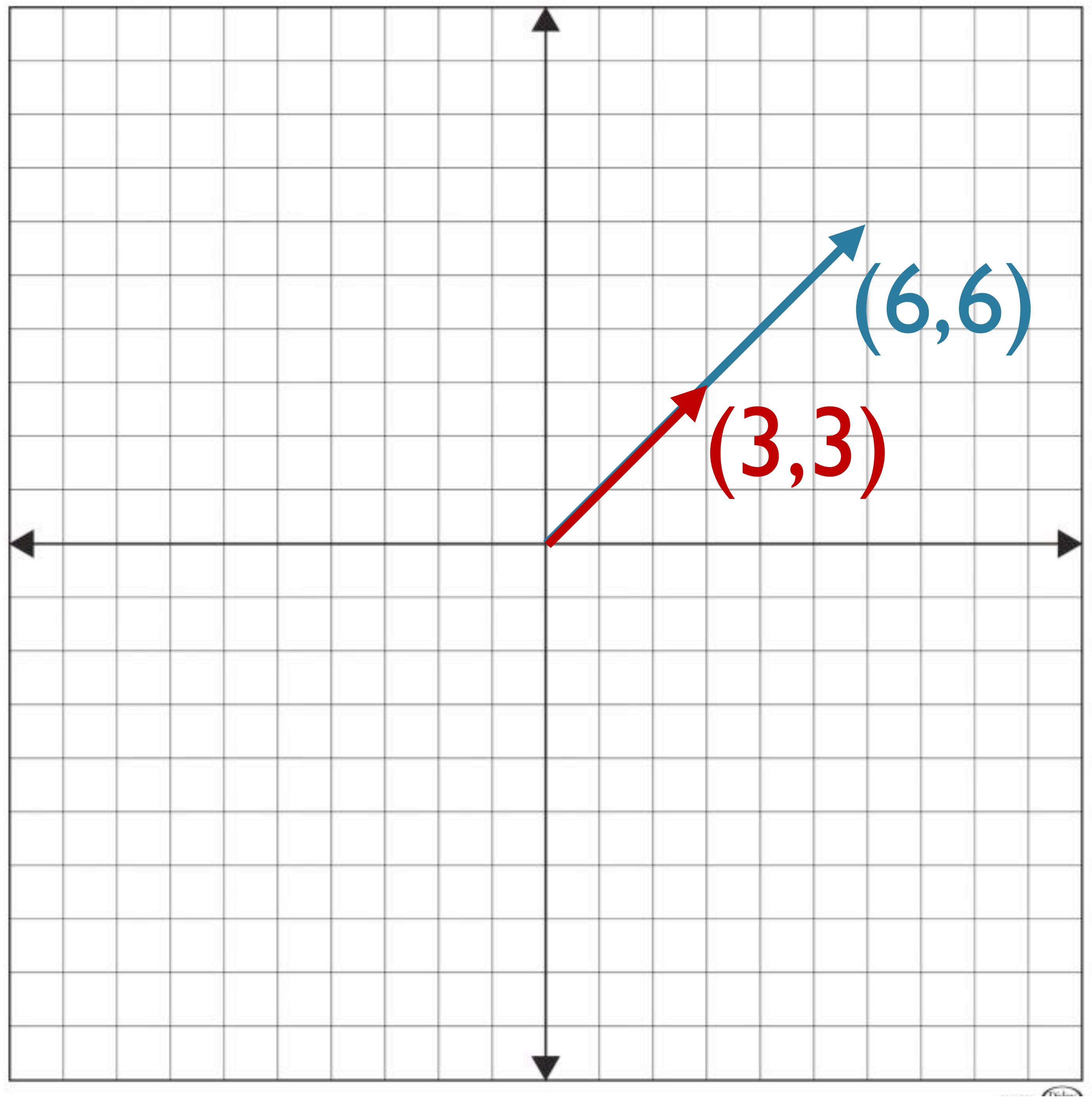

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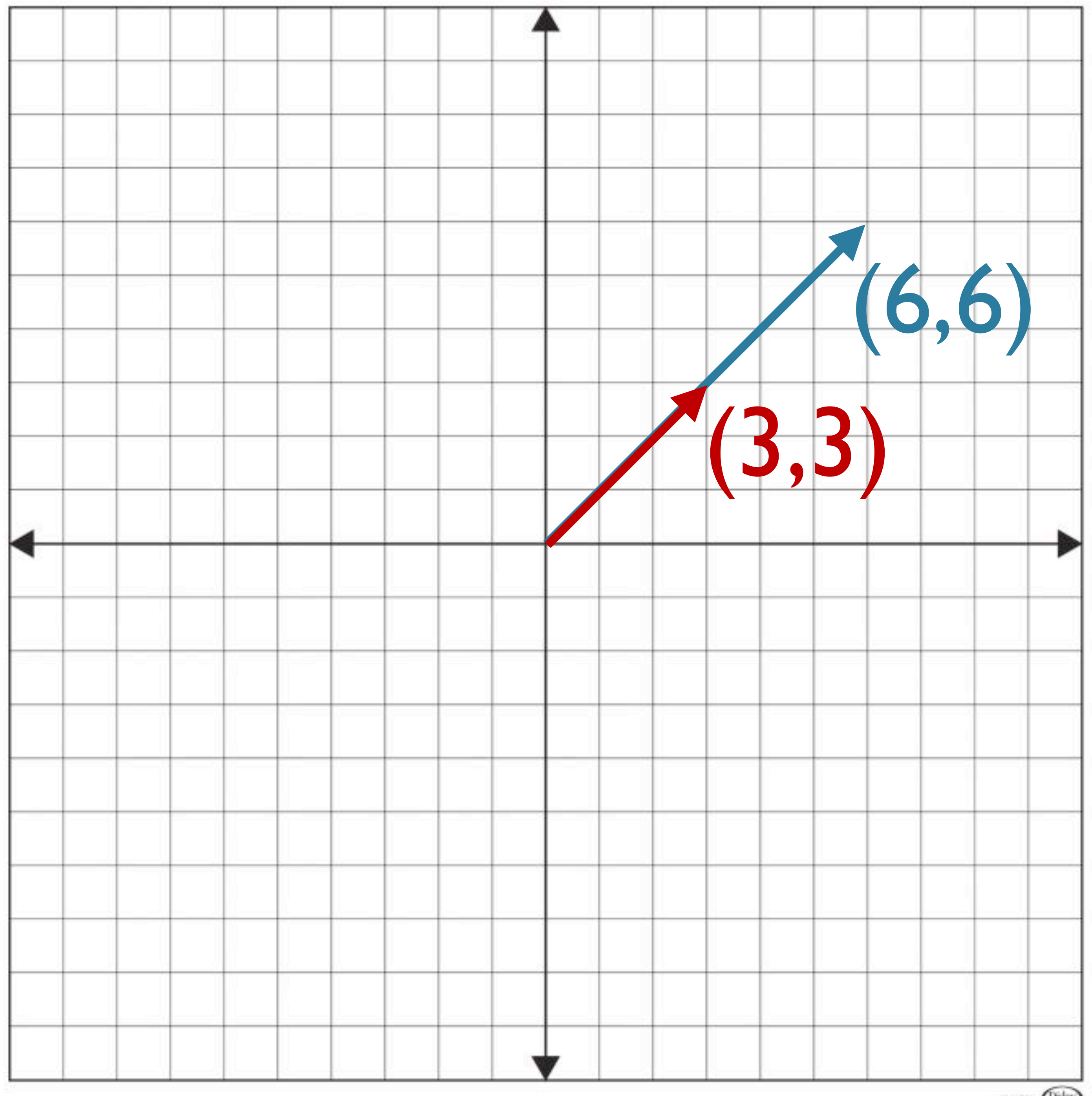

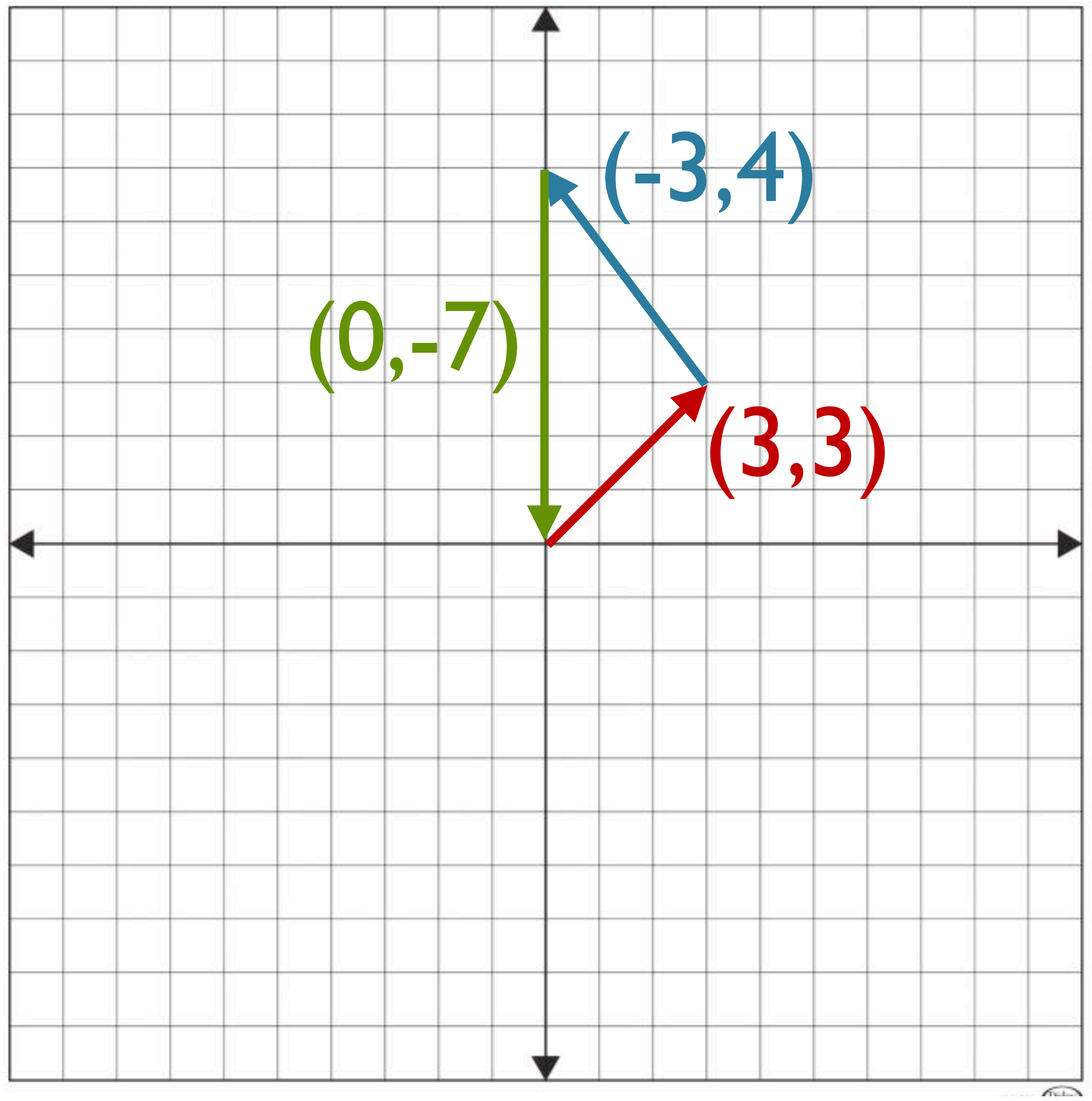

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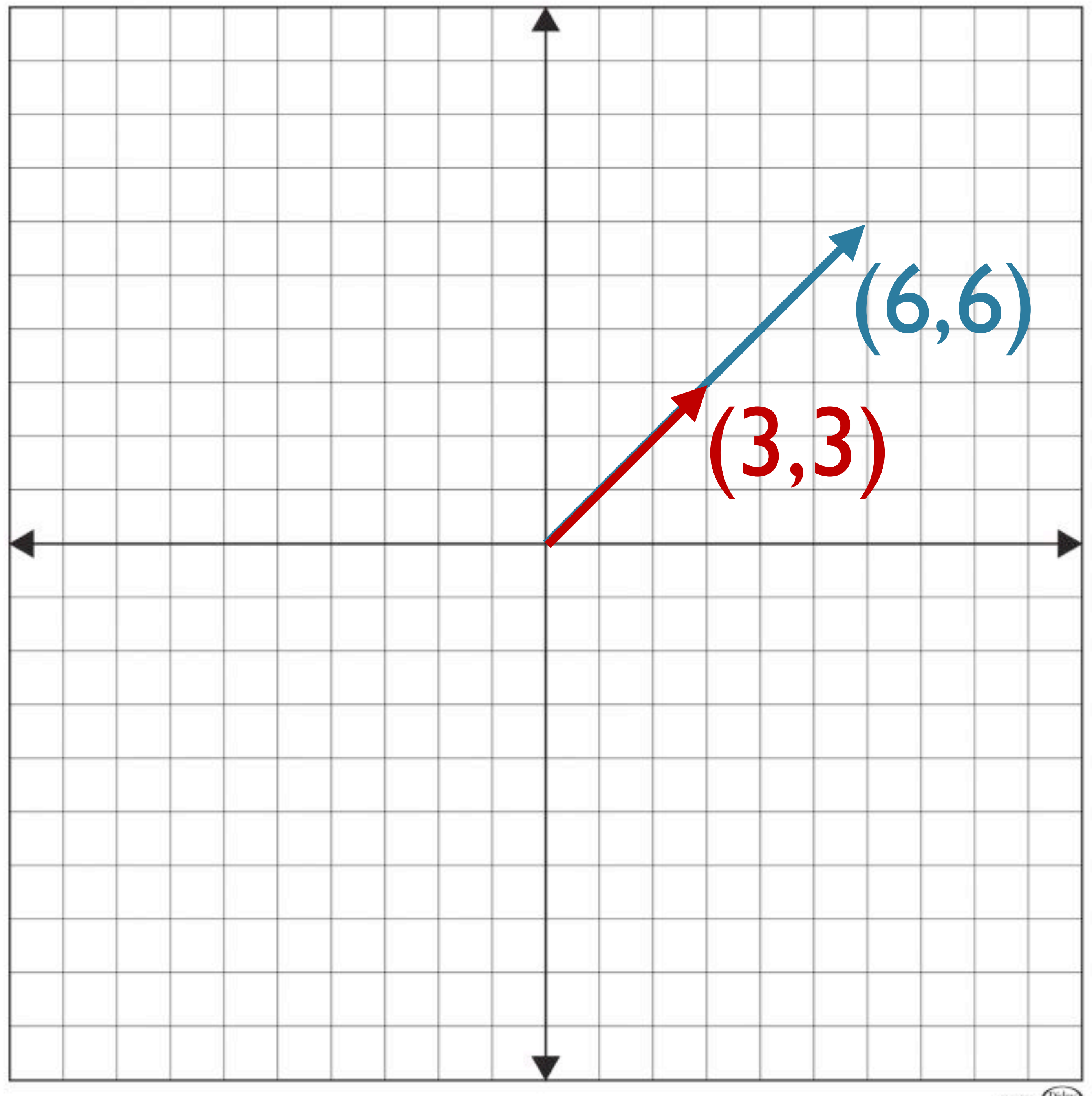

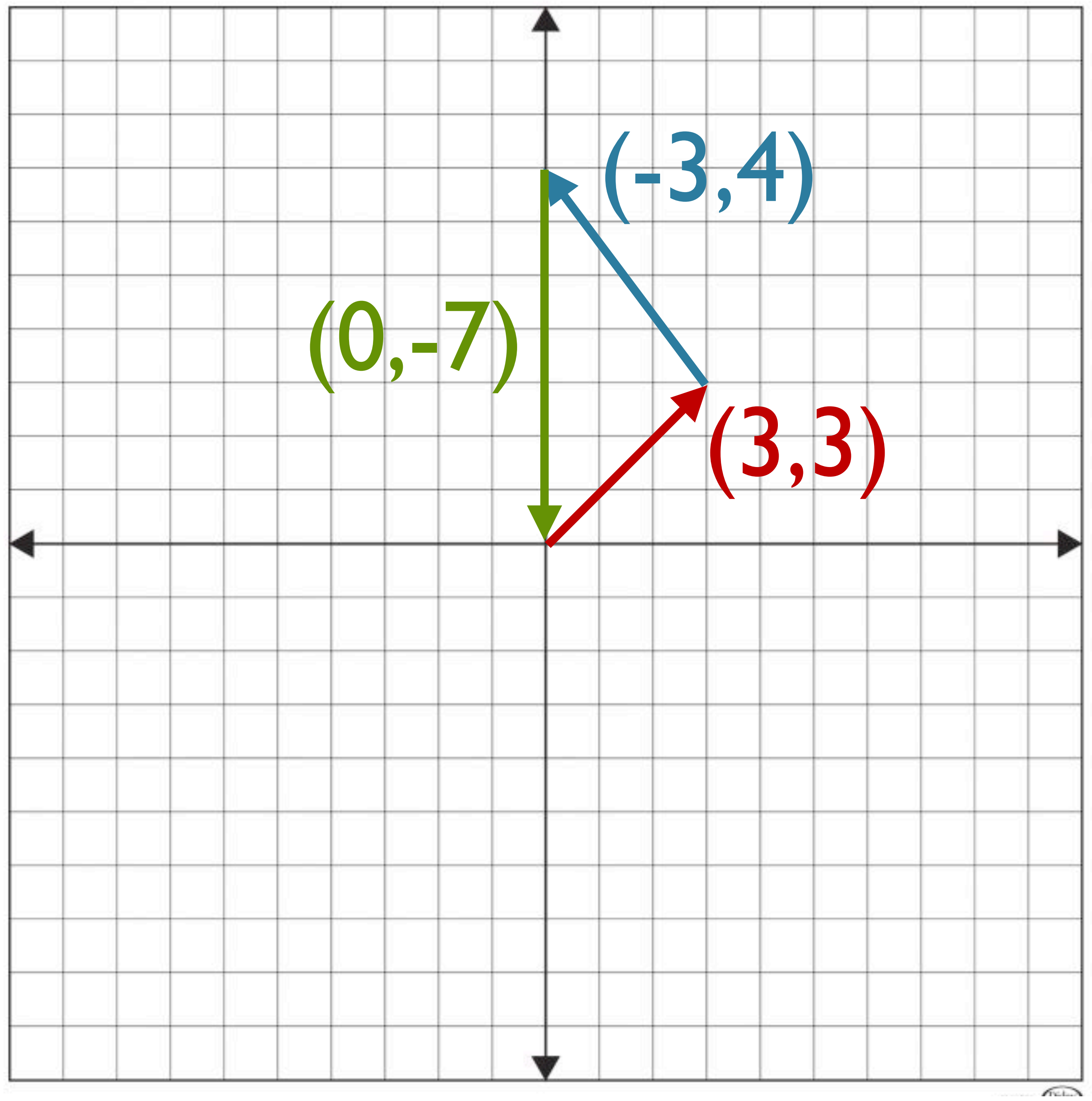

(this is what adding vectors looks like)

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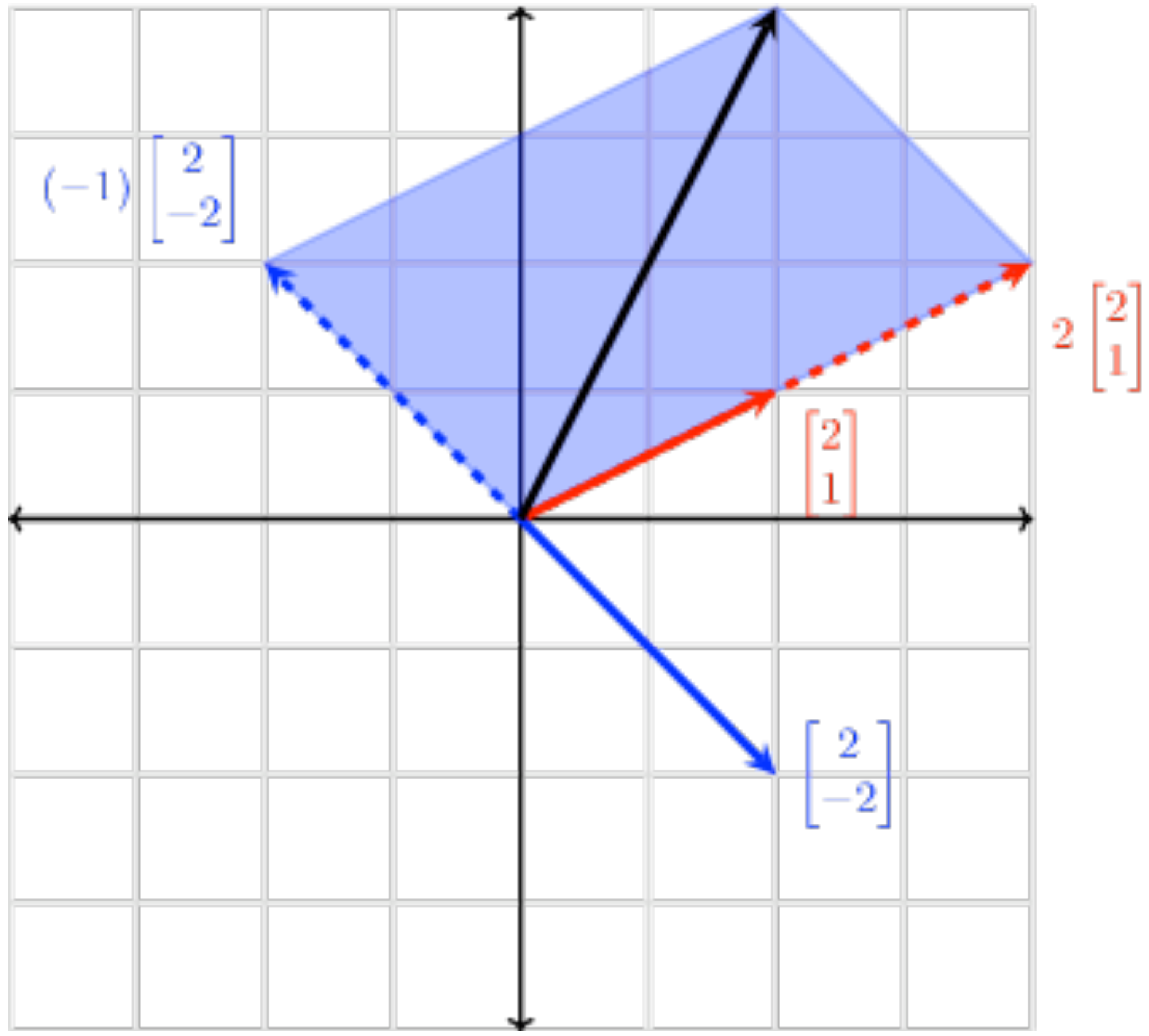

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  - Ex: **a** and **b** above span a **2-D plane in** R^3

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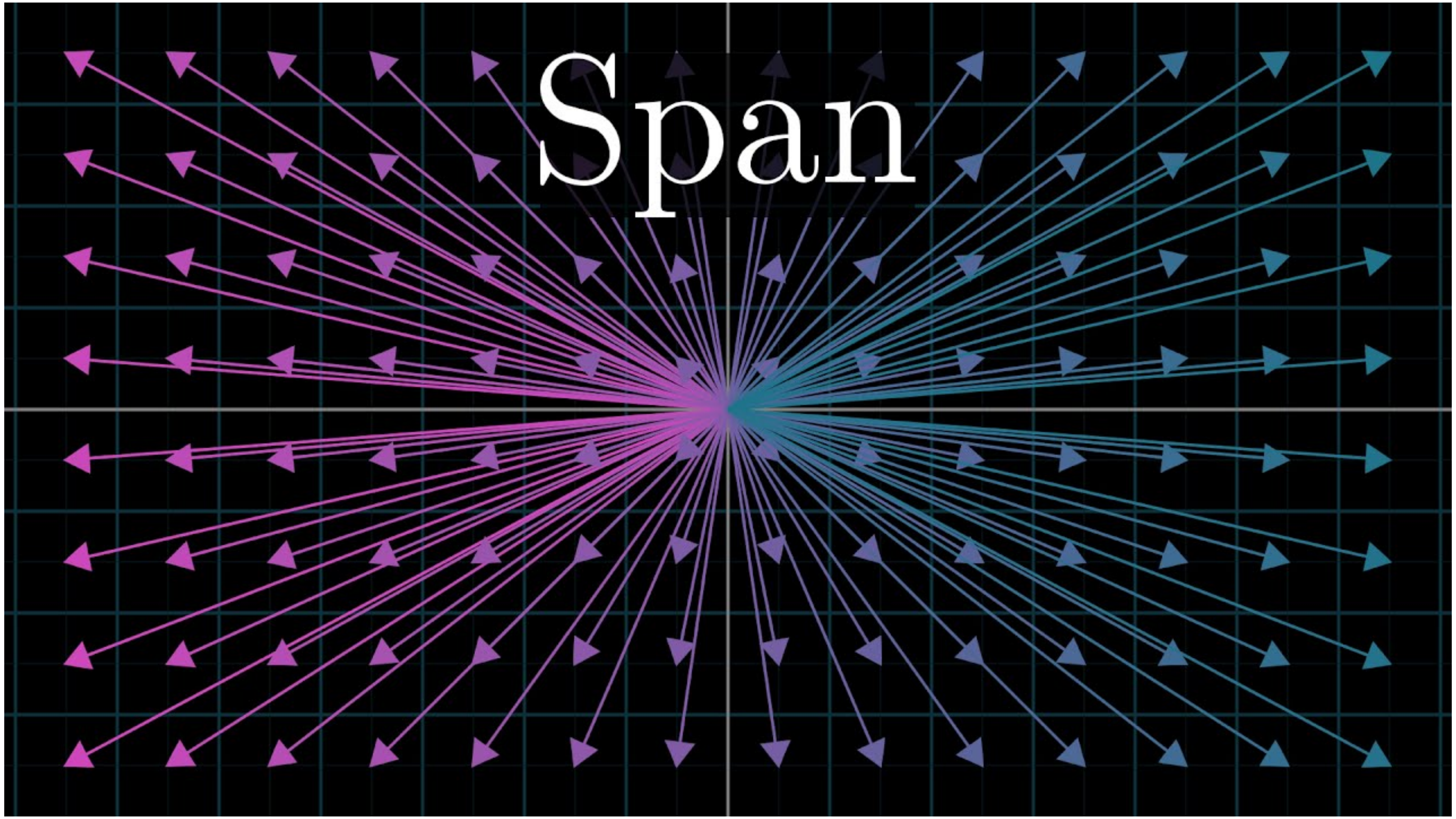
Span Video



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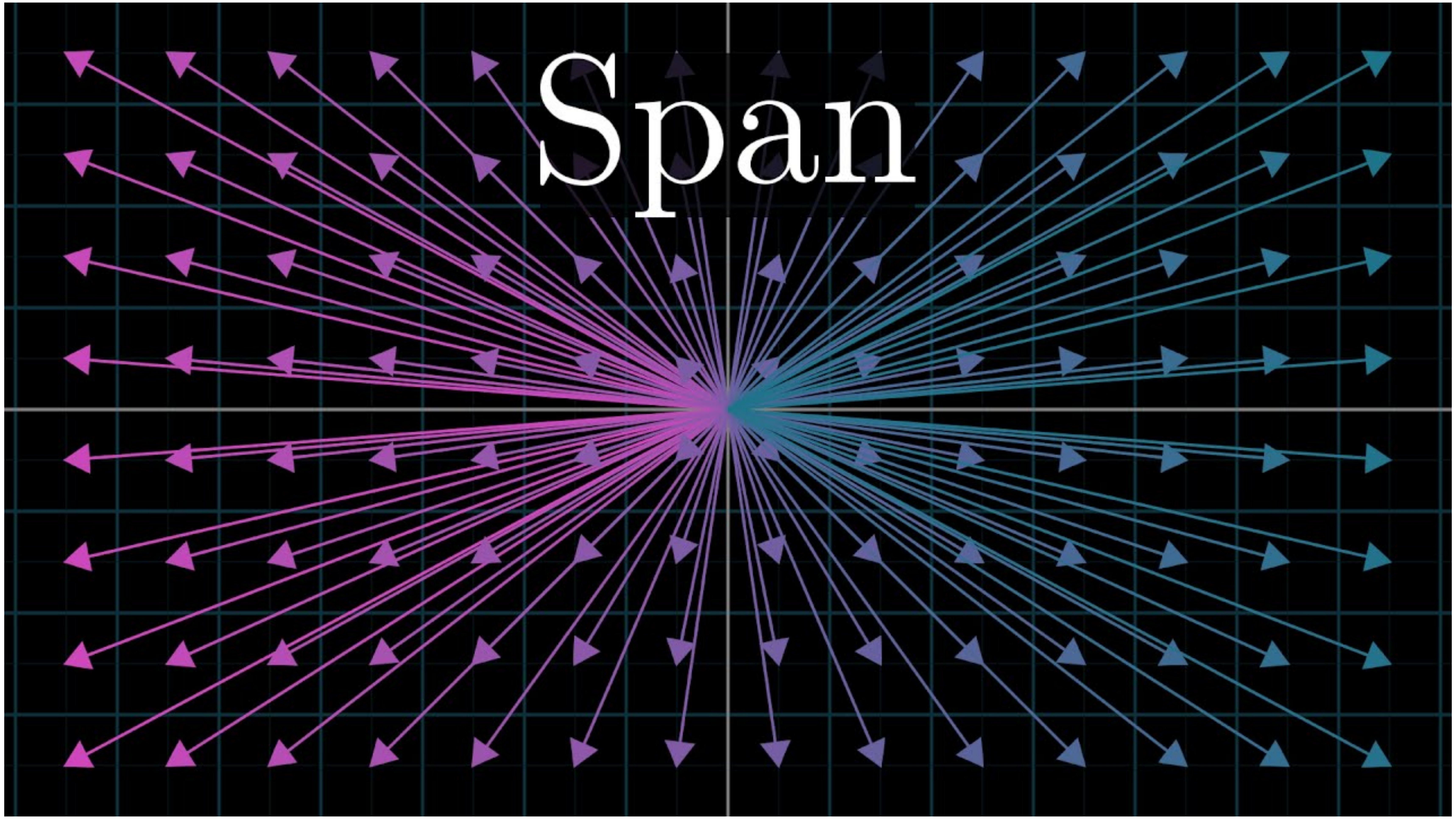

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Quick reminder: Dot Product

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(vectors need to be the same length)

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$$Ax = ?$$

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The Traditional Way

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \text{—} \\ \text{—} \\ \text{—} \end{bmatrix}$$

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The Traditional Way

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ \boxed{3 \quad 6 \quad 9} \end{bmatrix} \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 12 \\ 15 \\ - \end{bmatrix}$$

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- Alternative way to think about this multiplication
  - The matrix consists of **column vectors**

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  - **However**, the function's range **may not span** R^M, unless it is **rank M**

Linear Transformations

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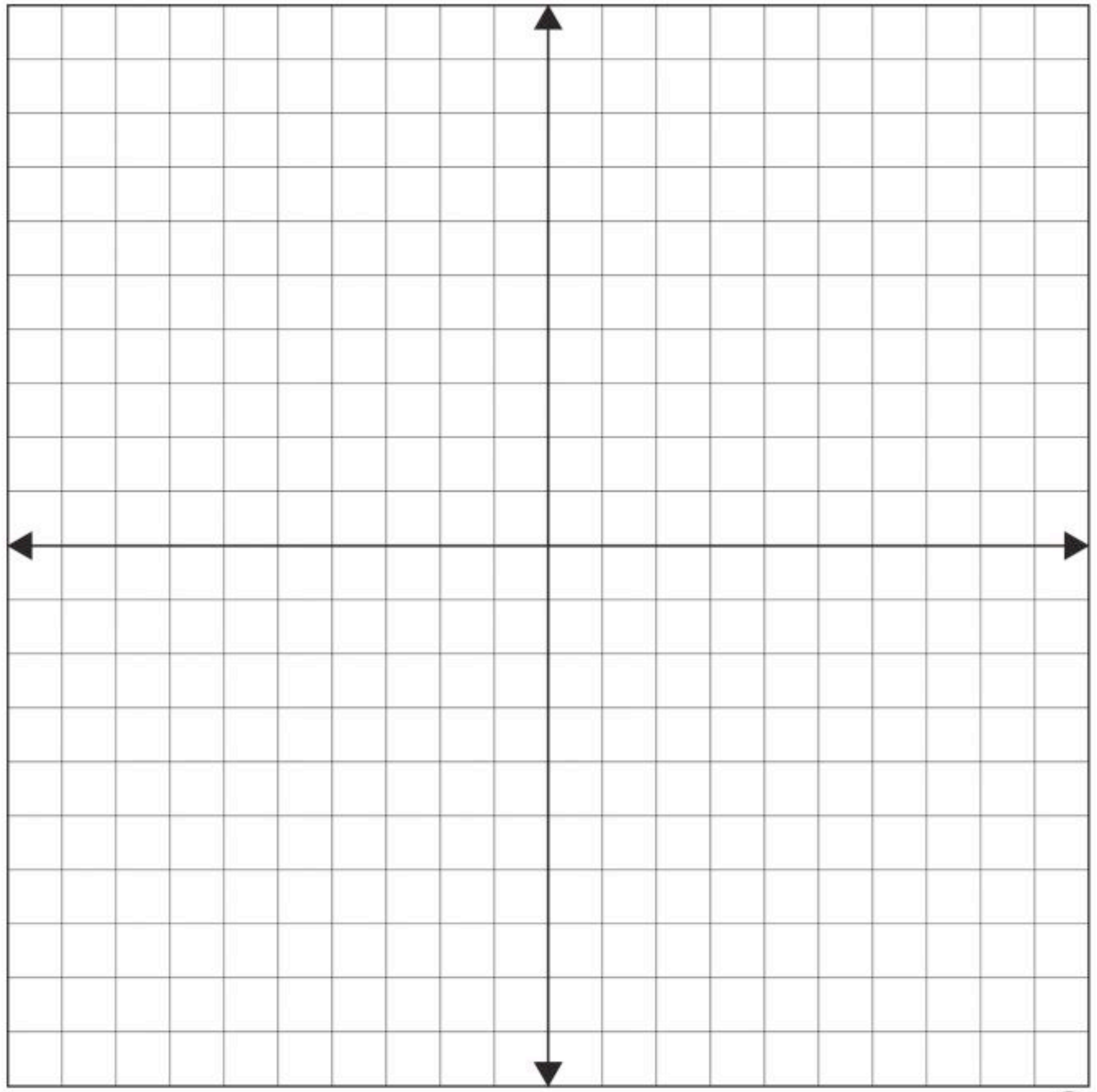
Identity Matrix as a Basis

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



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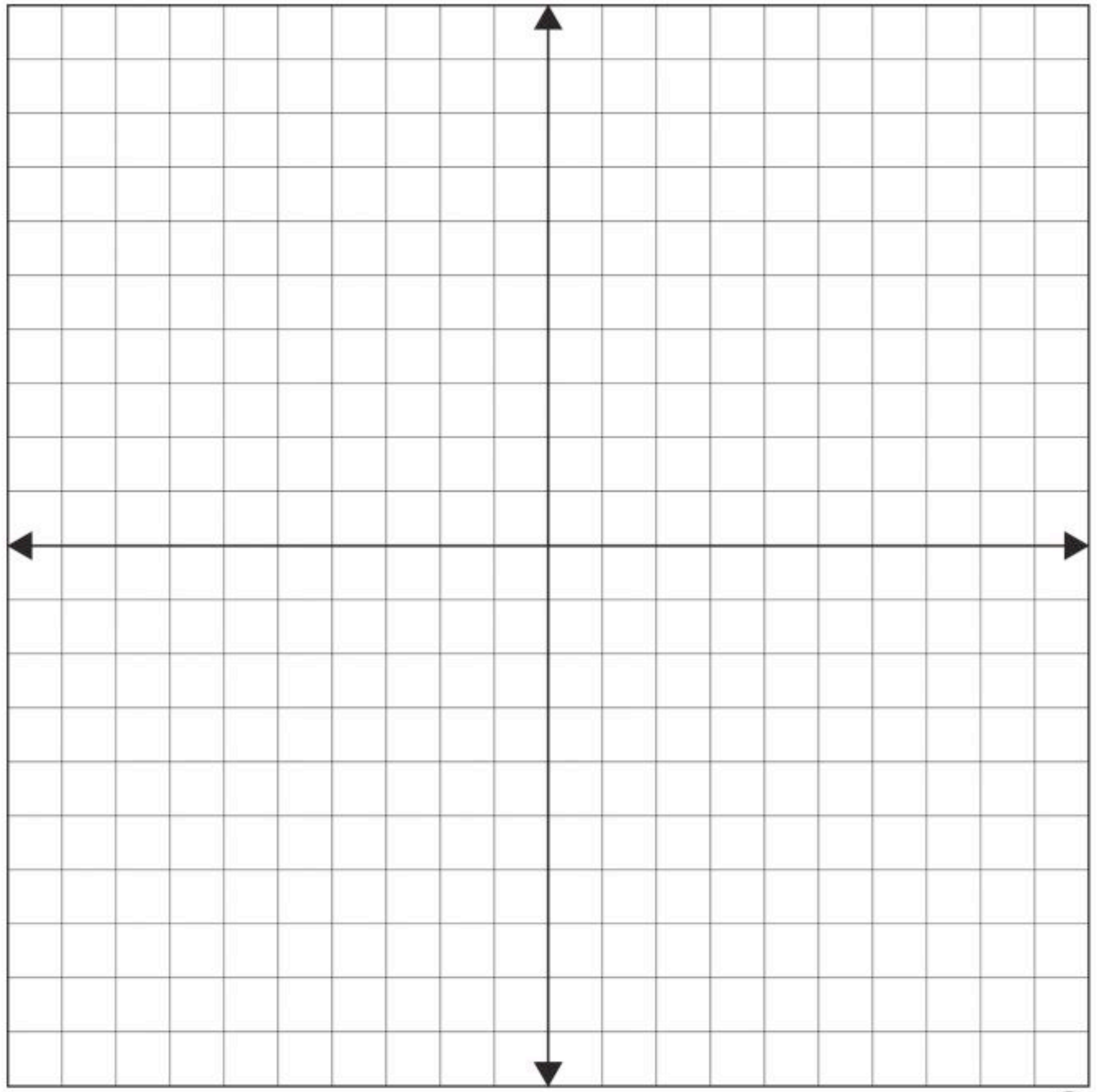

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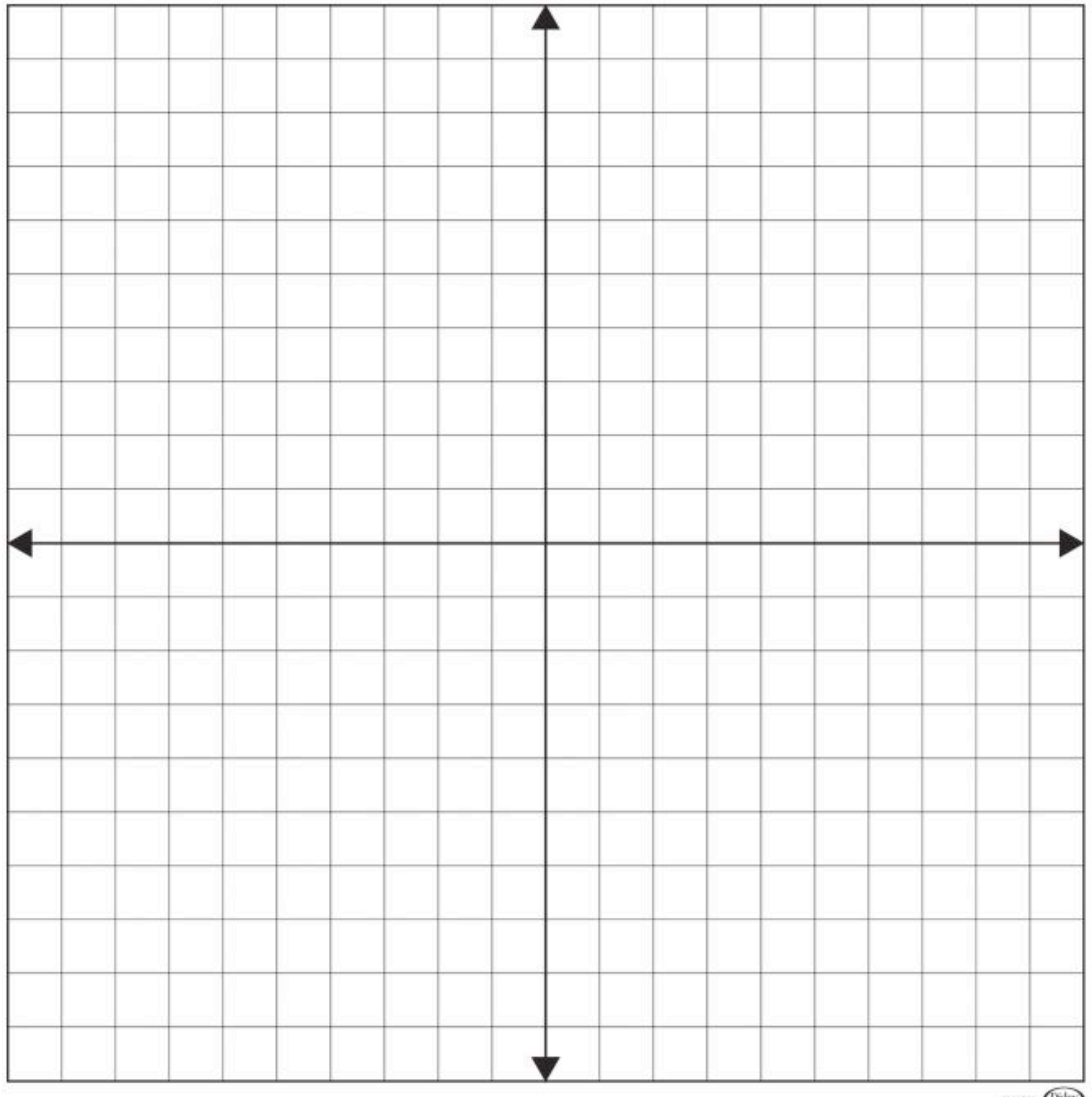

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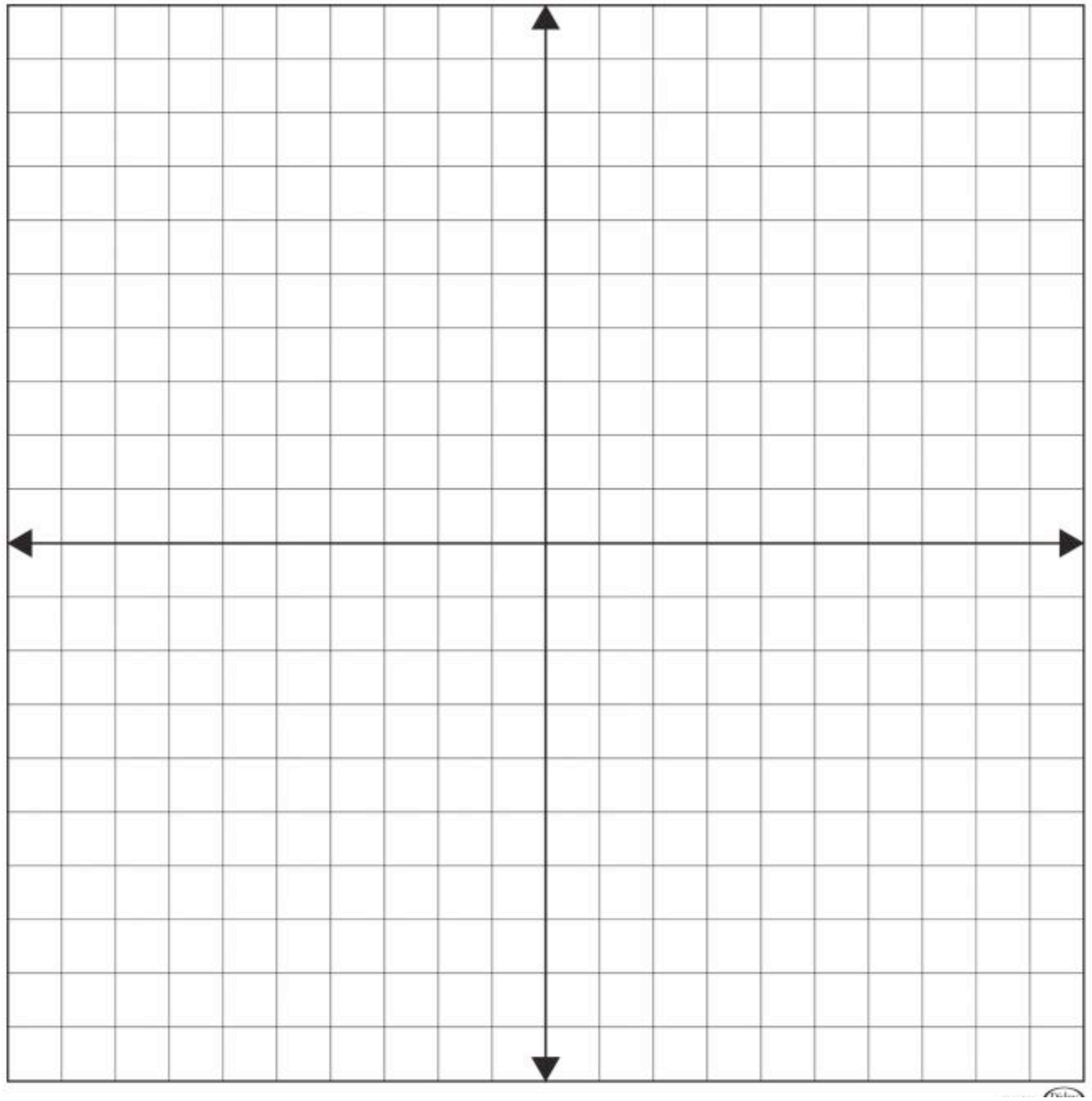

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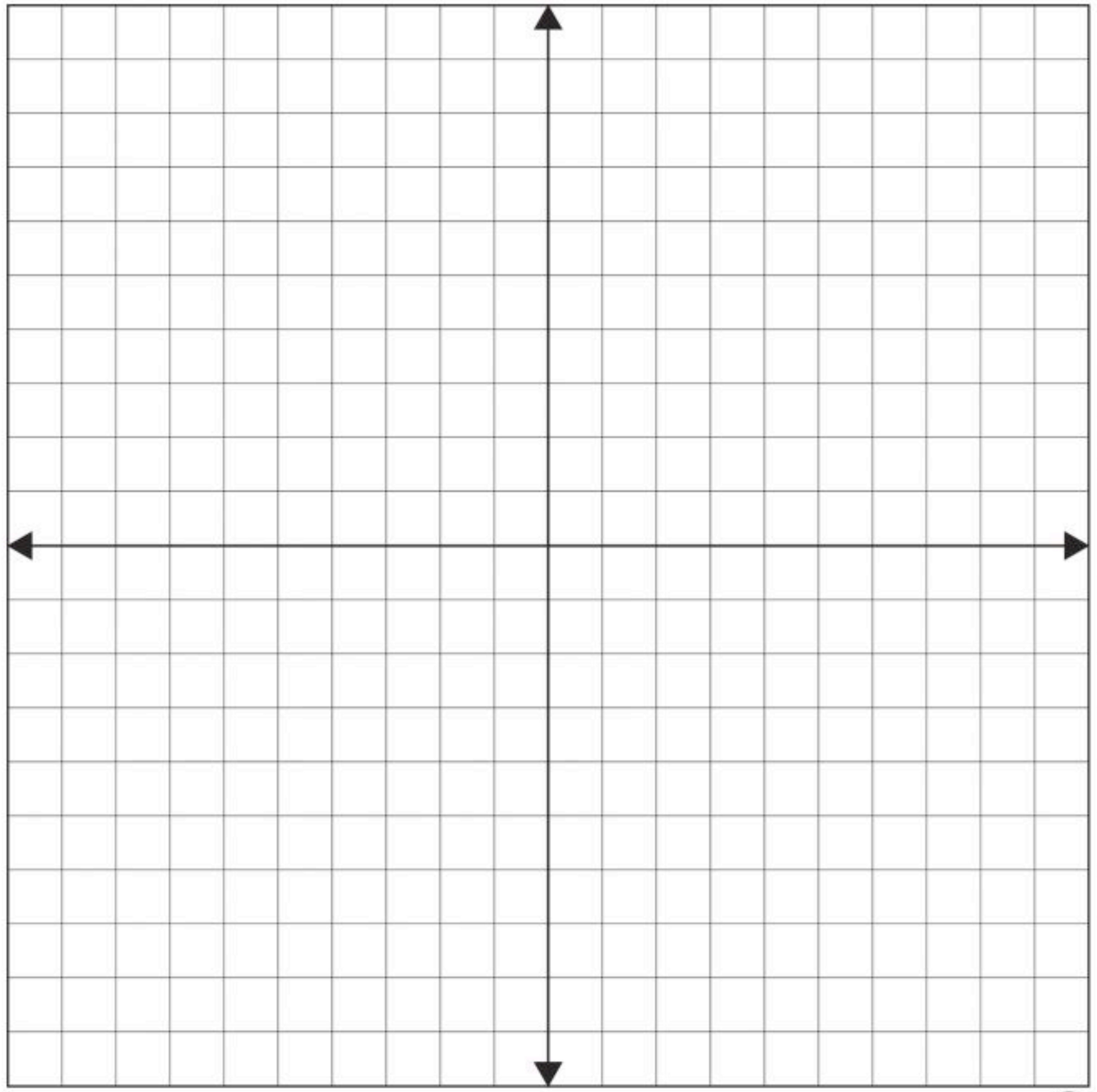

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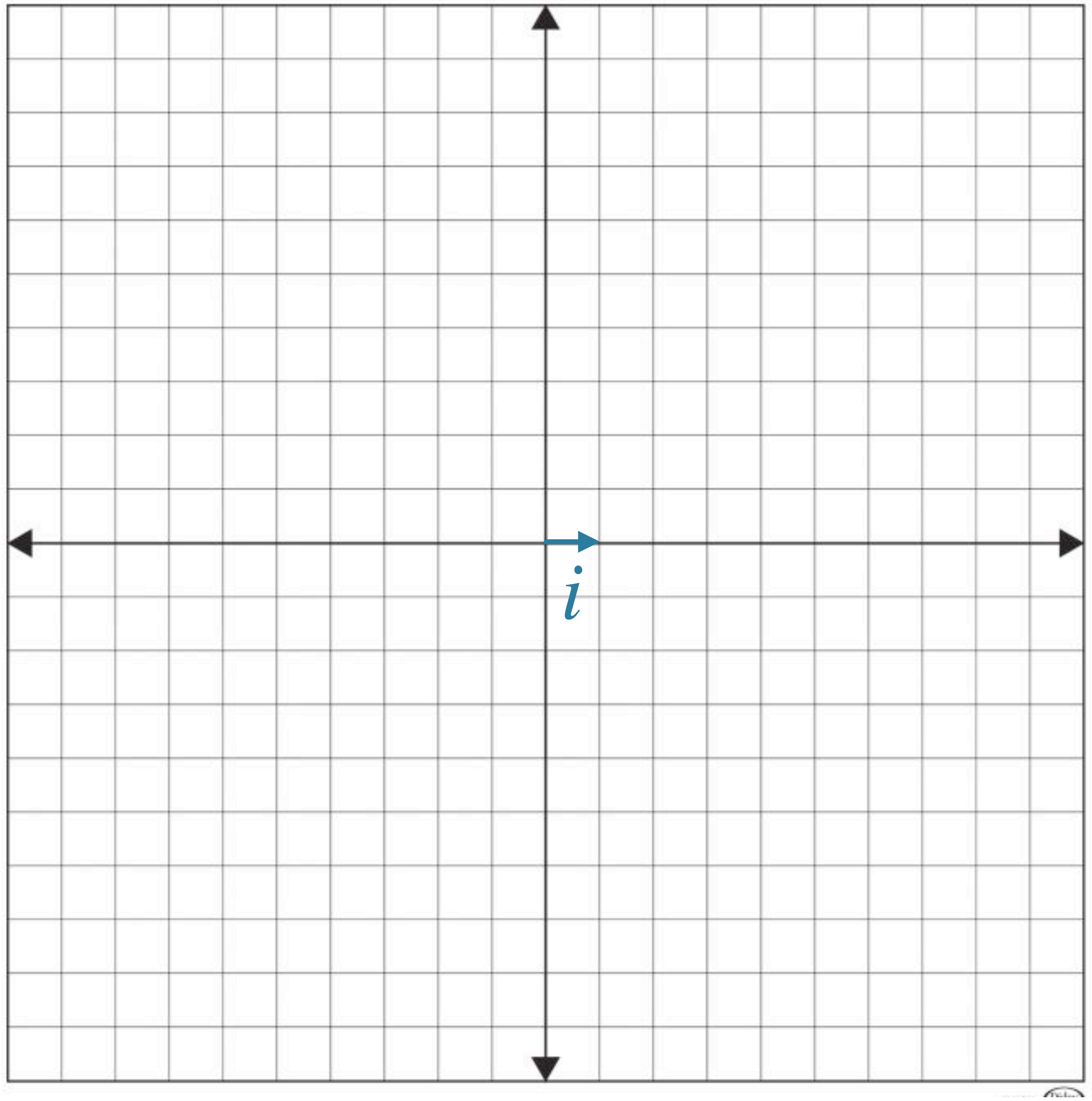

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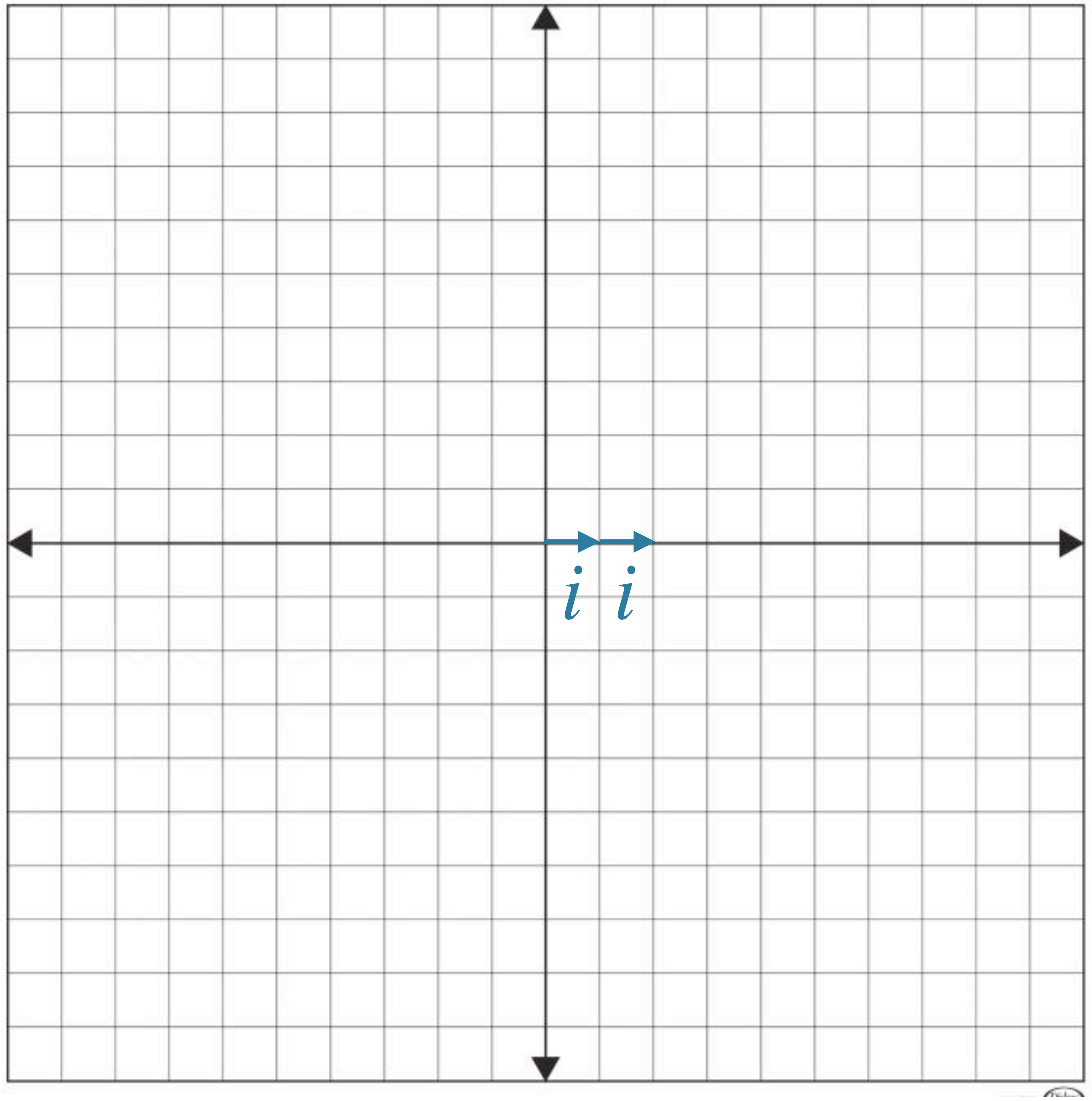

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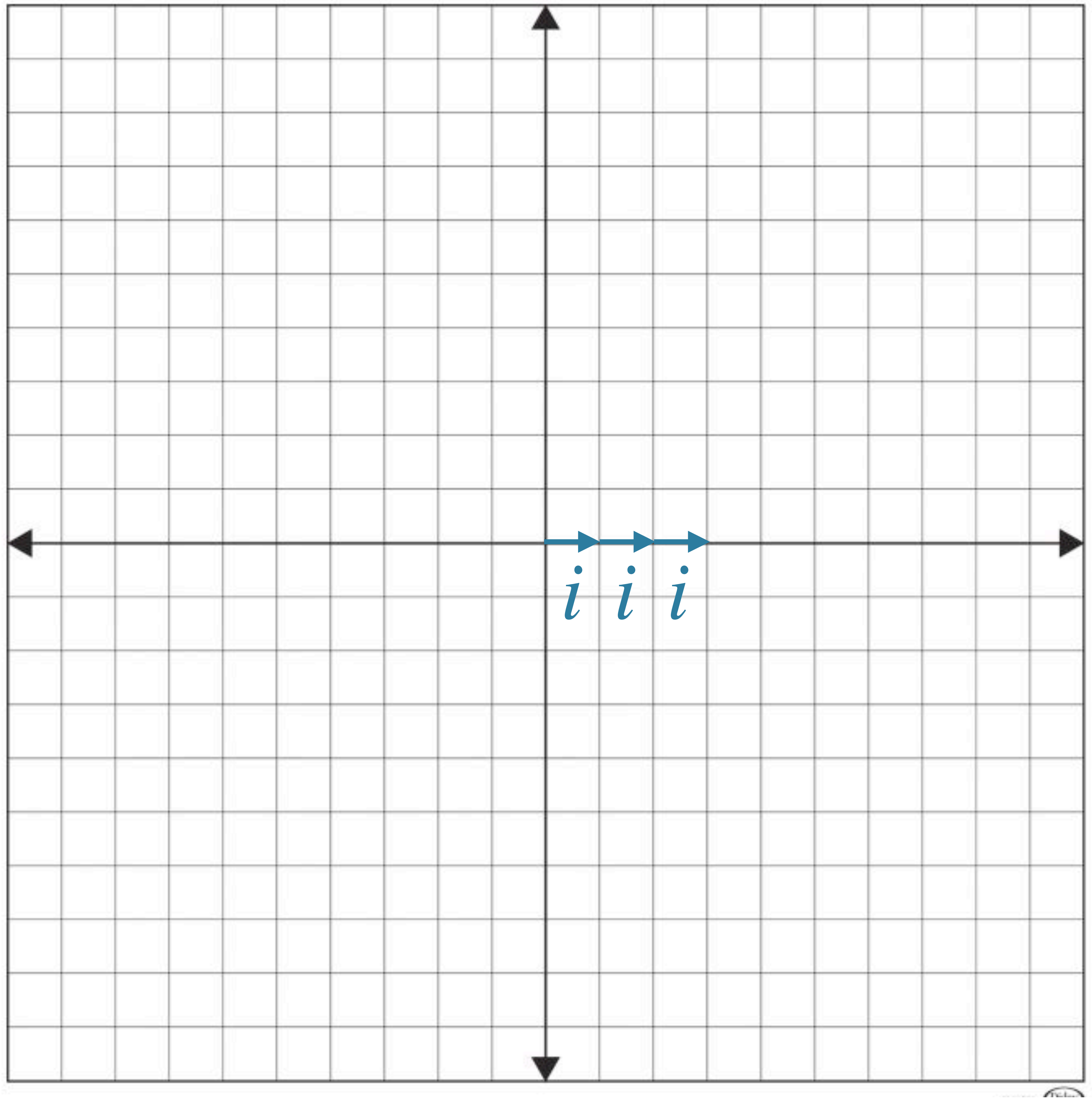

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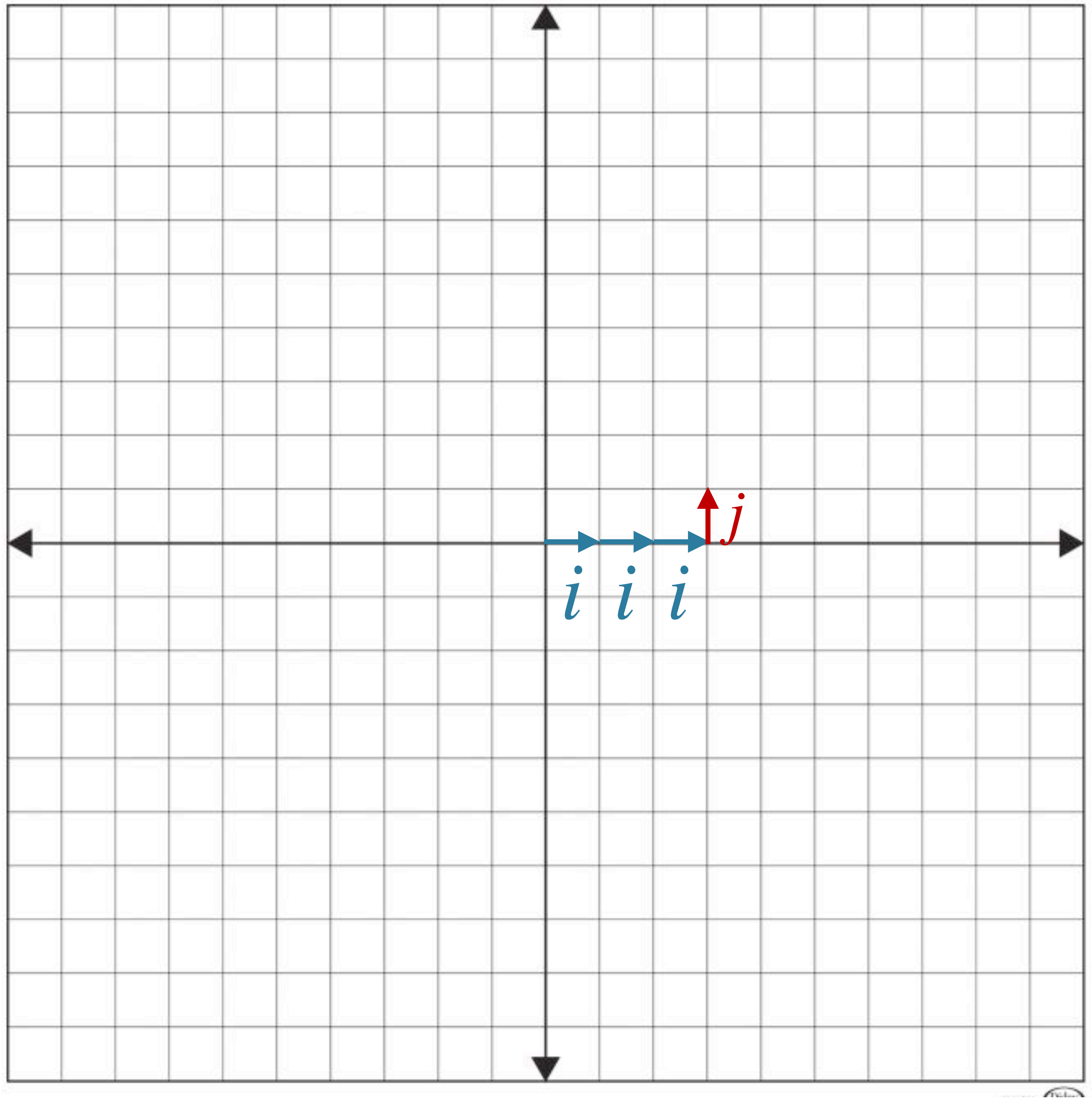

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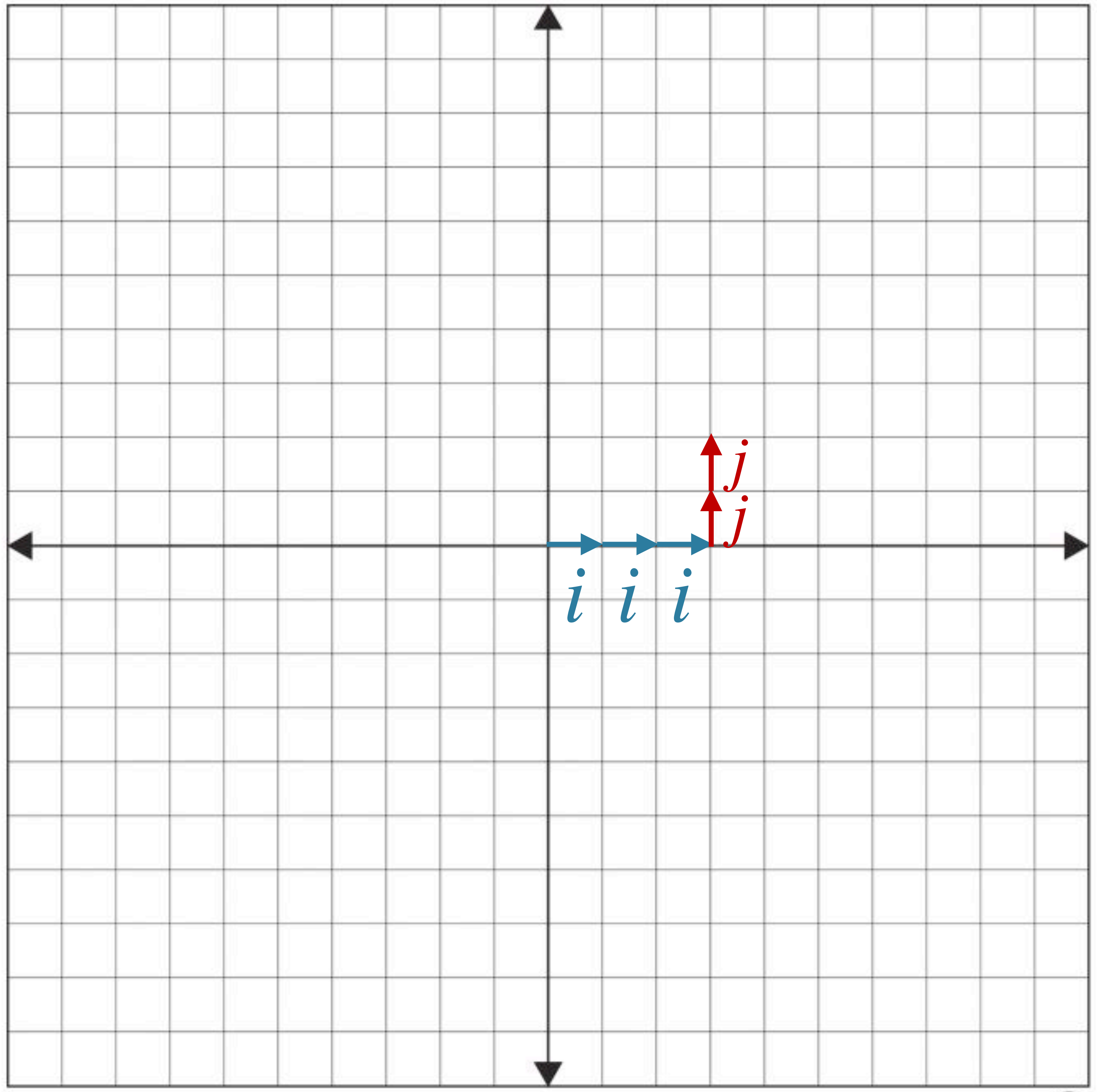

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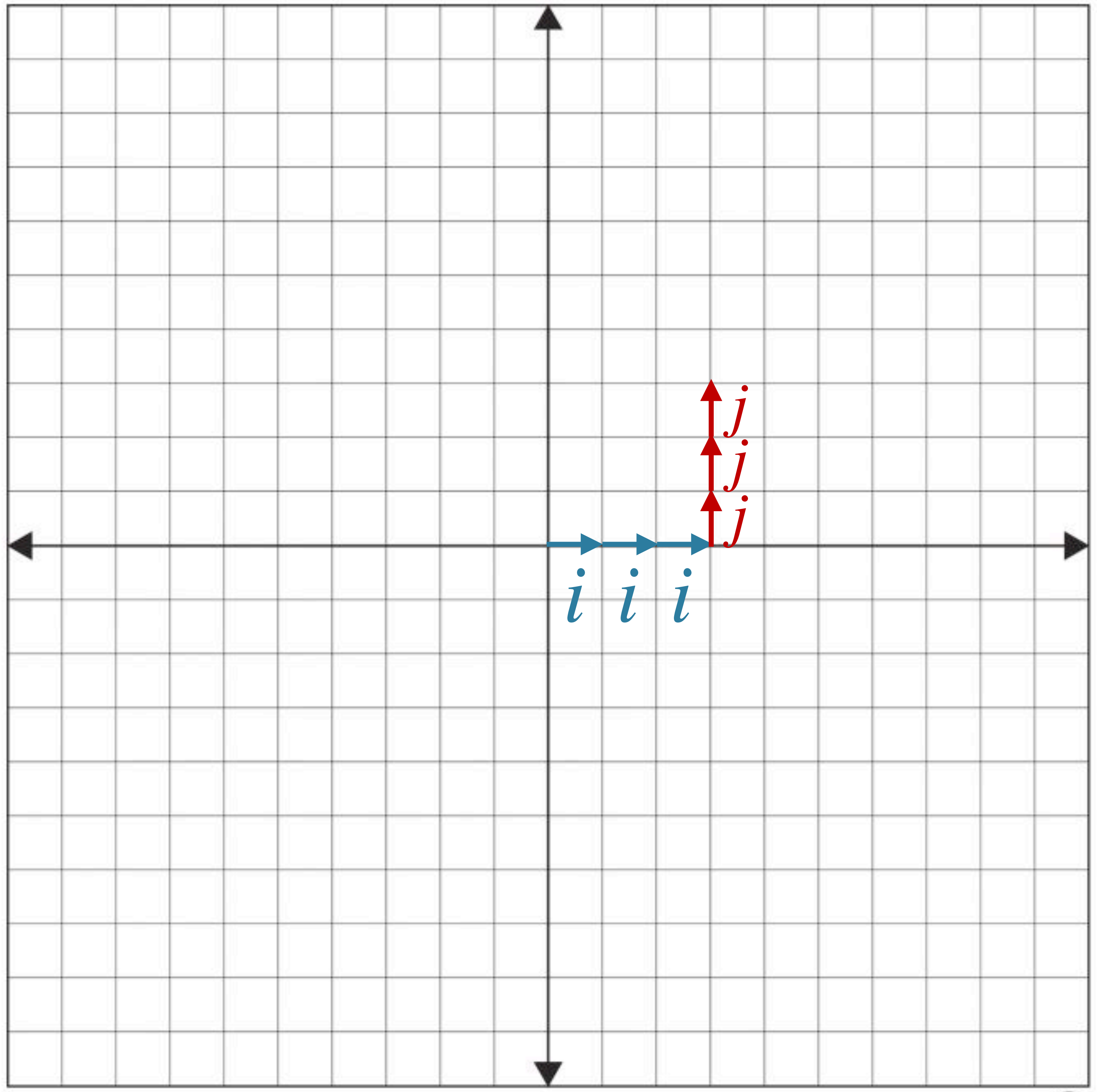

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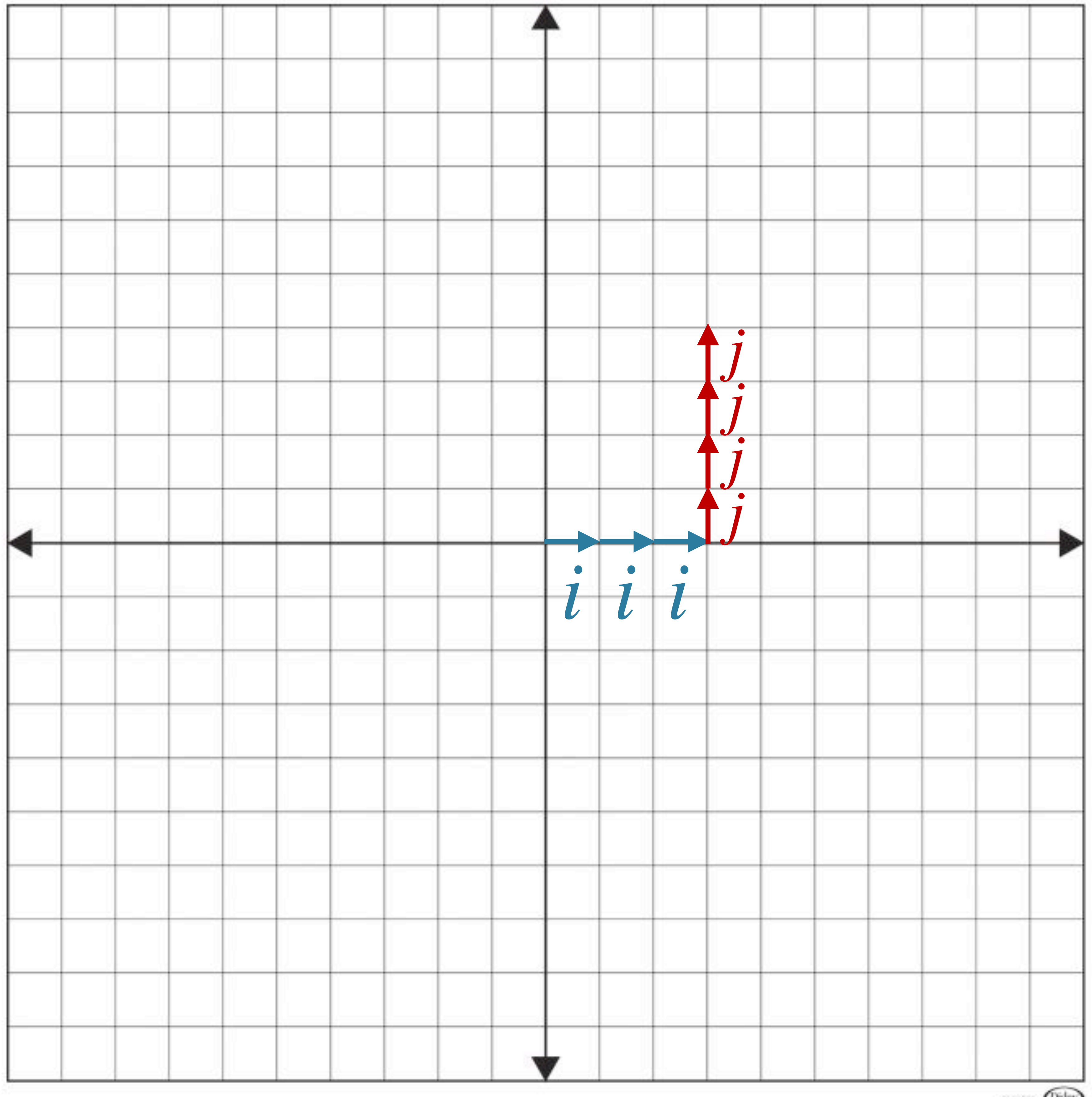

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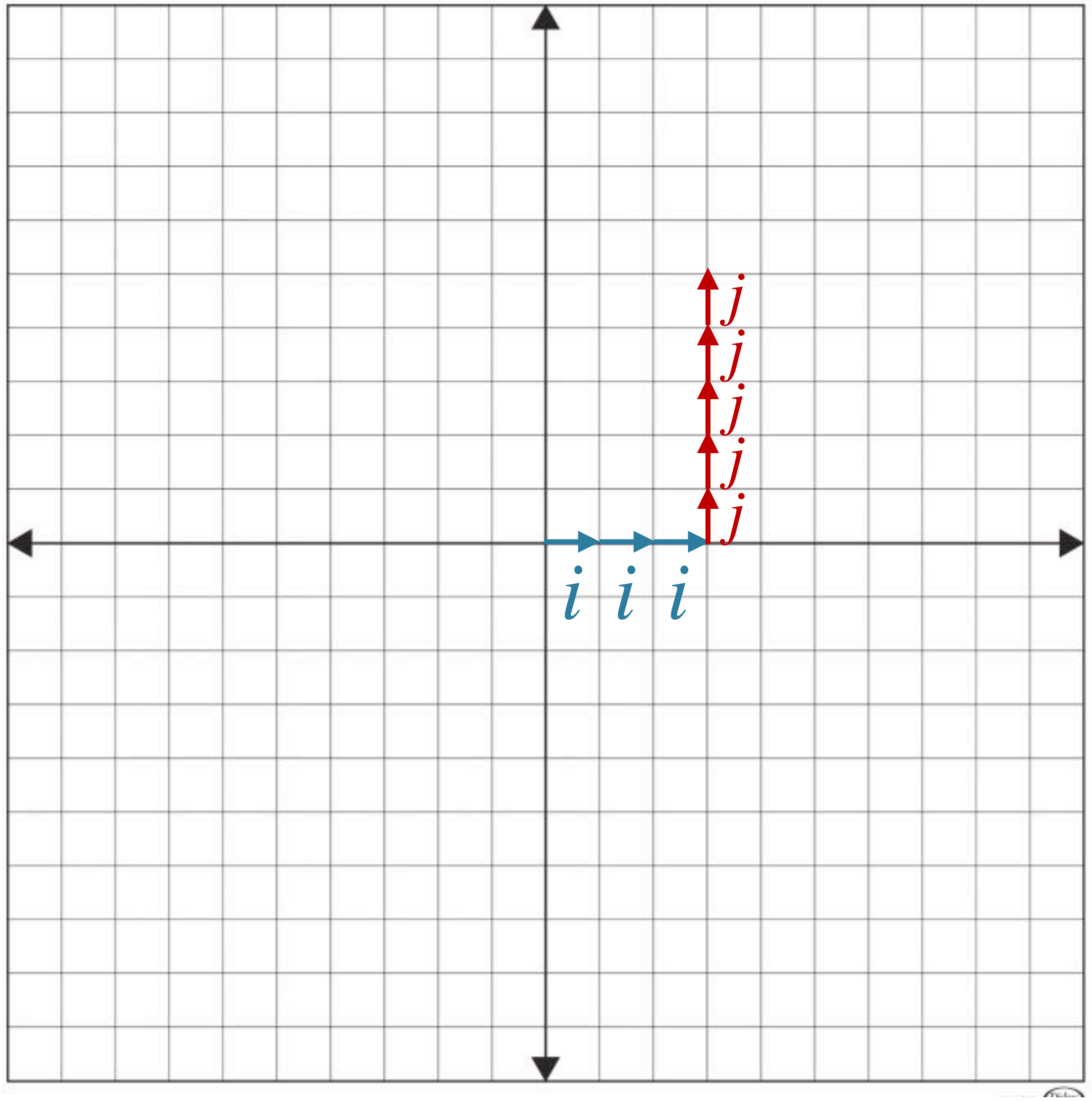

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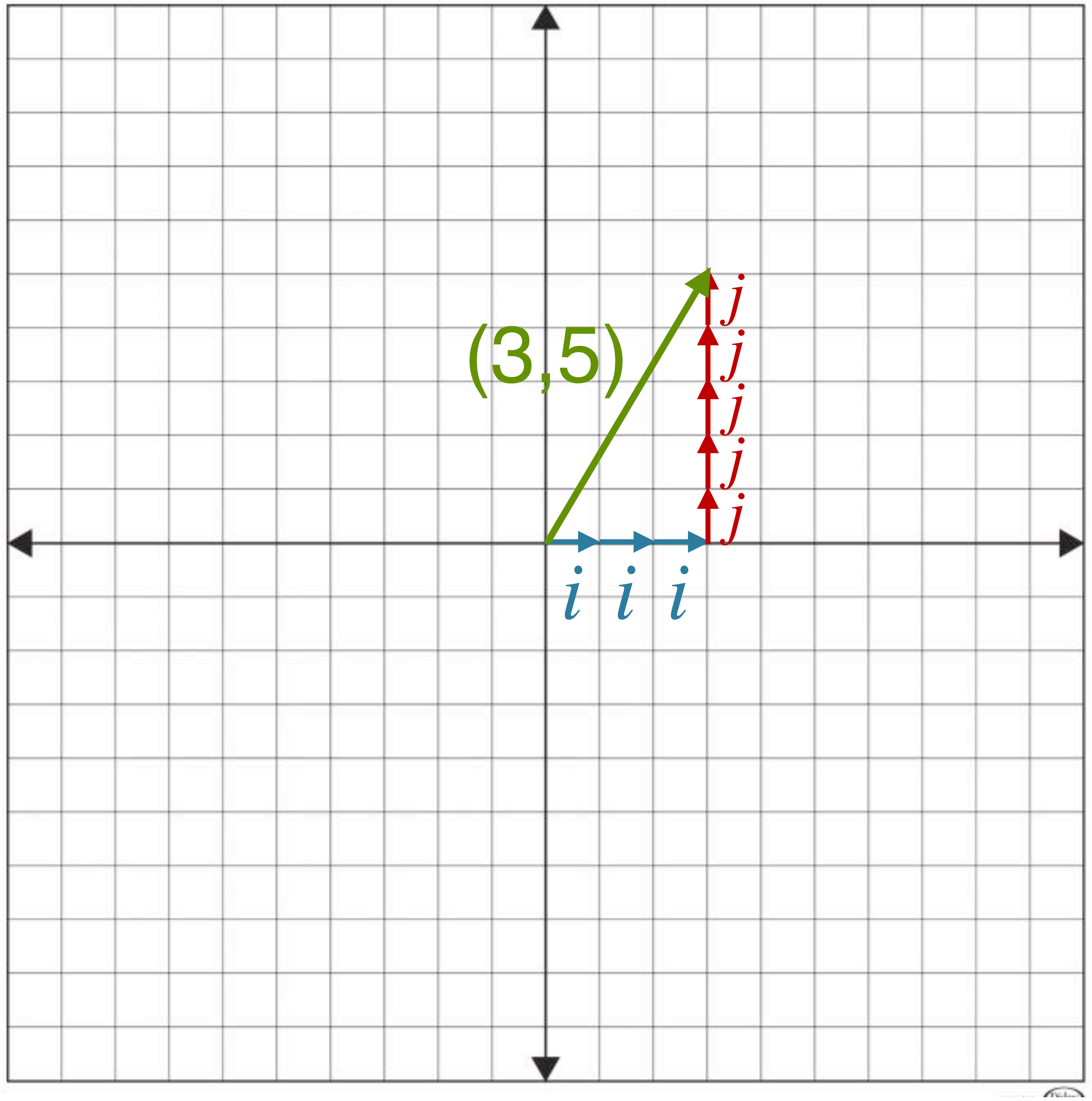

Linear Transformation

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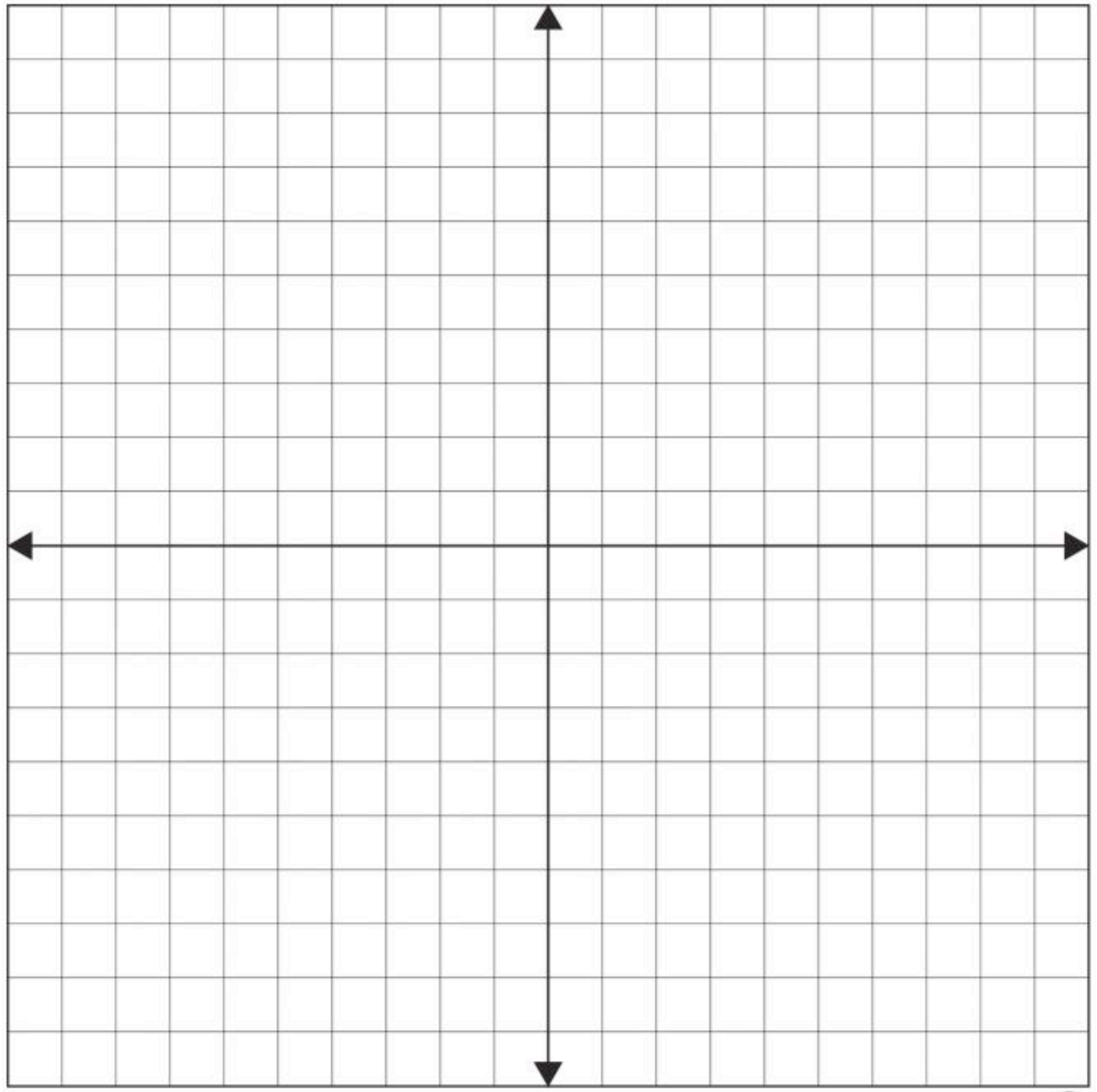

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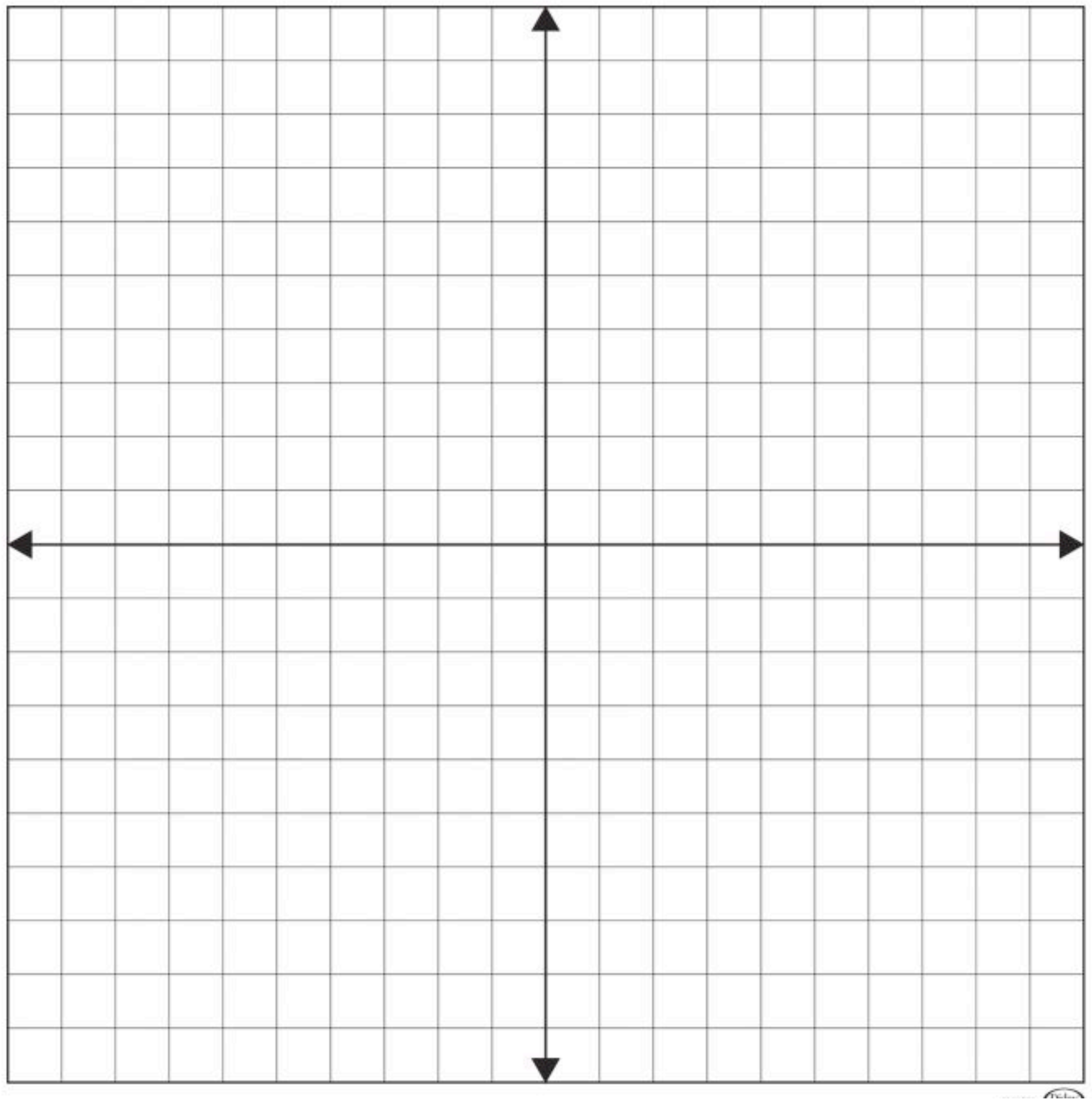

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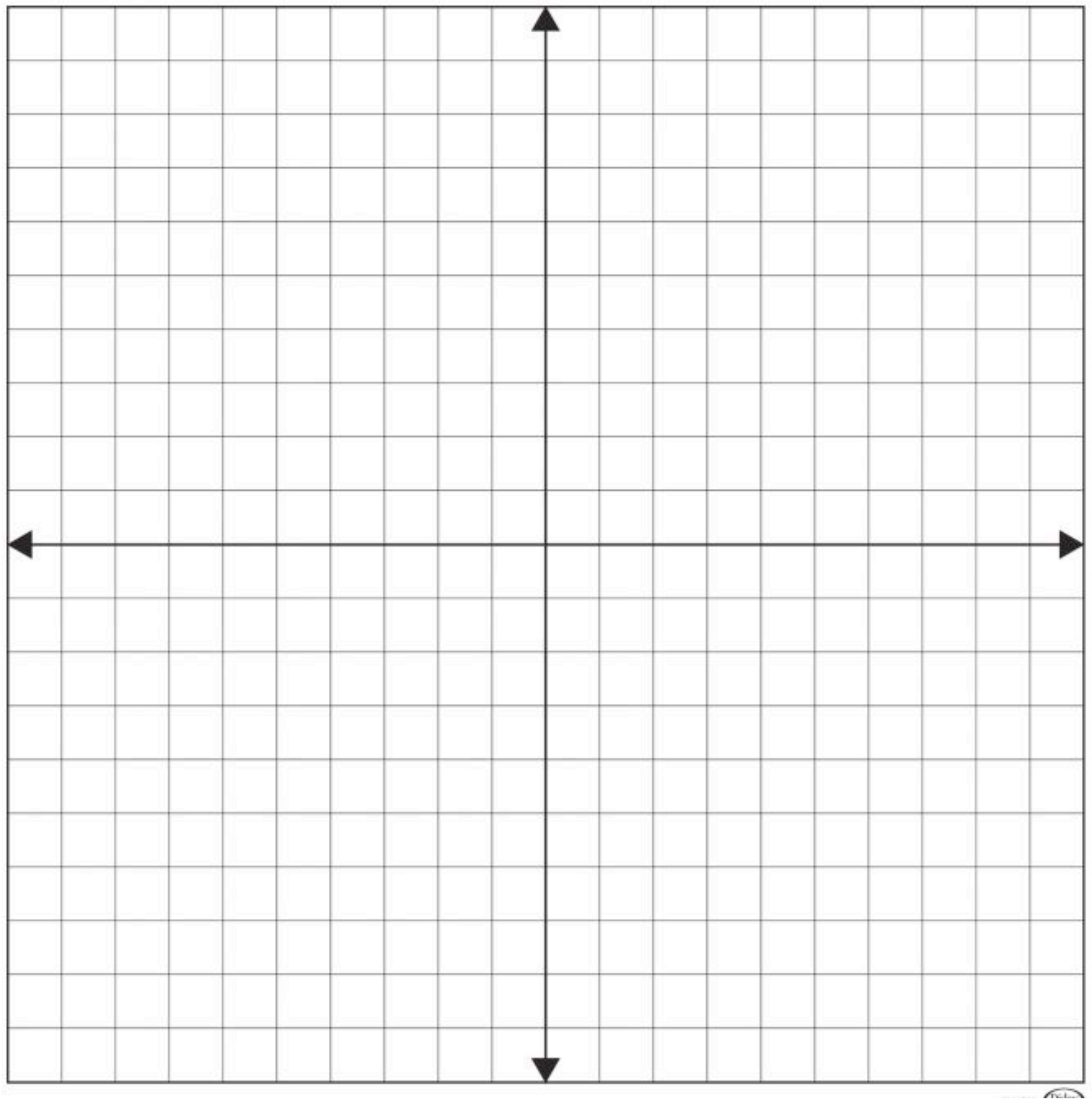

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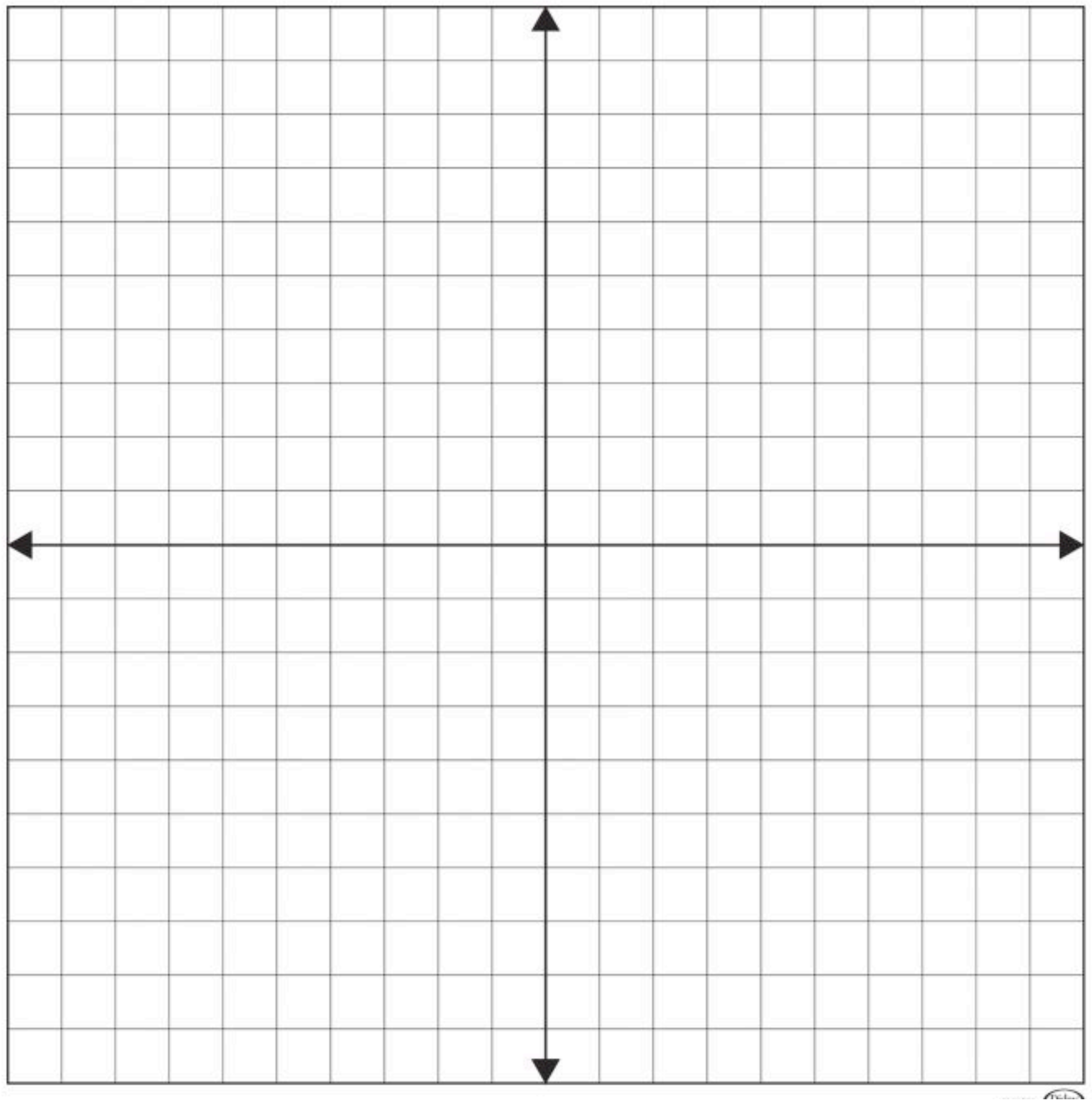

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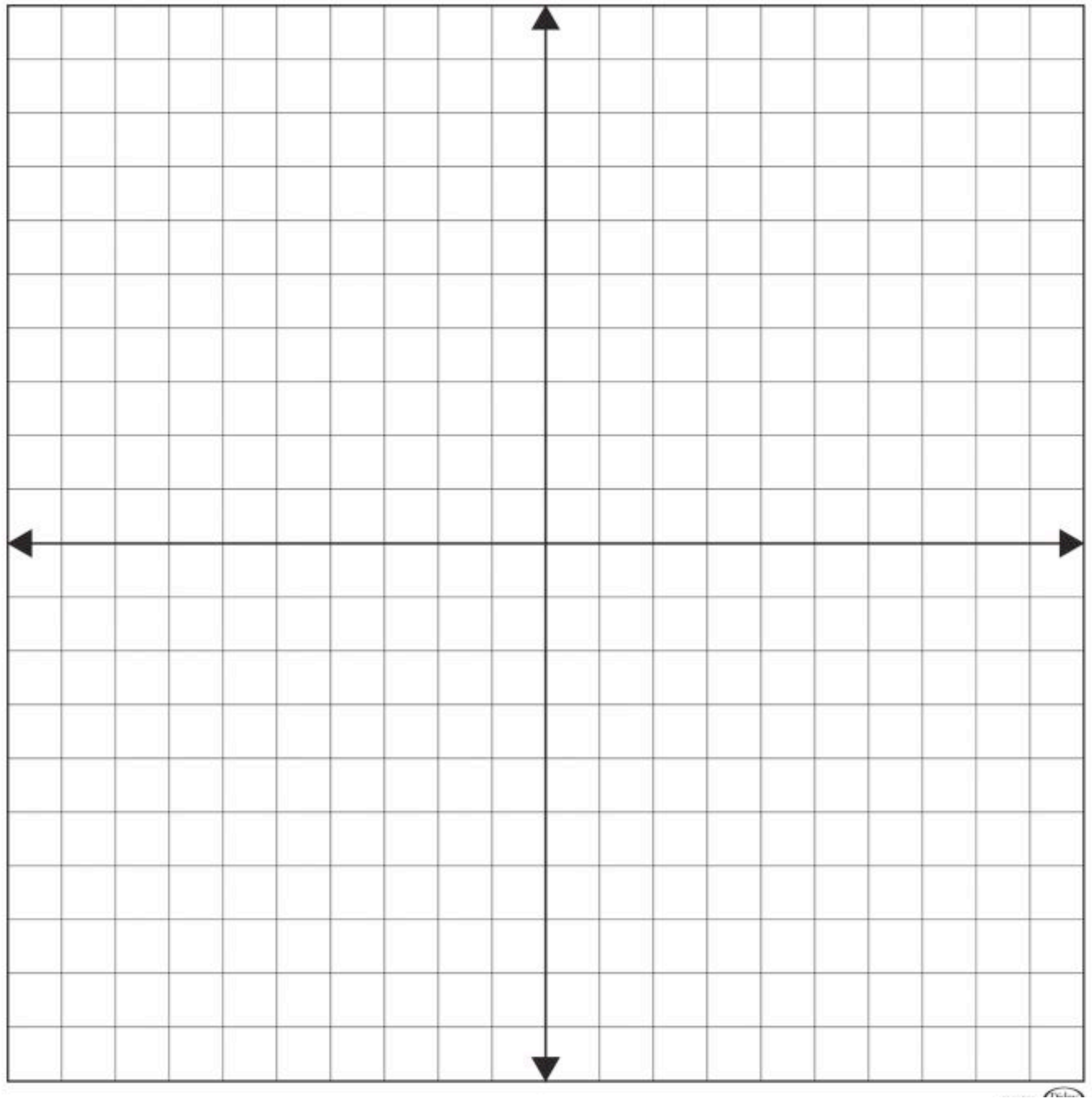

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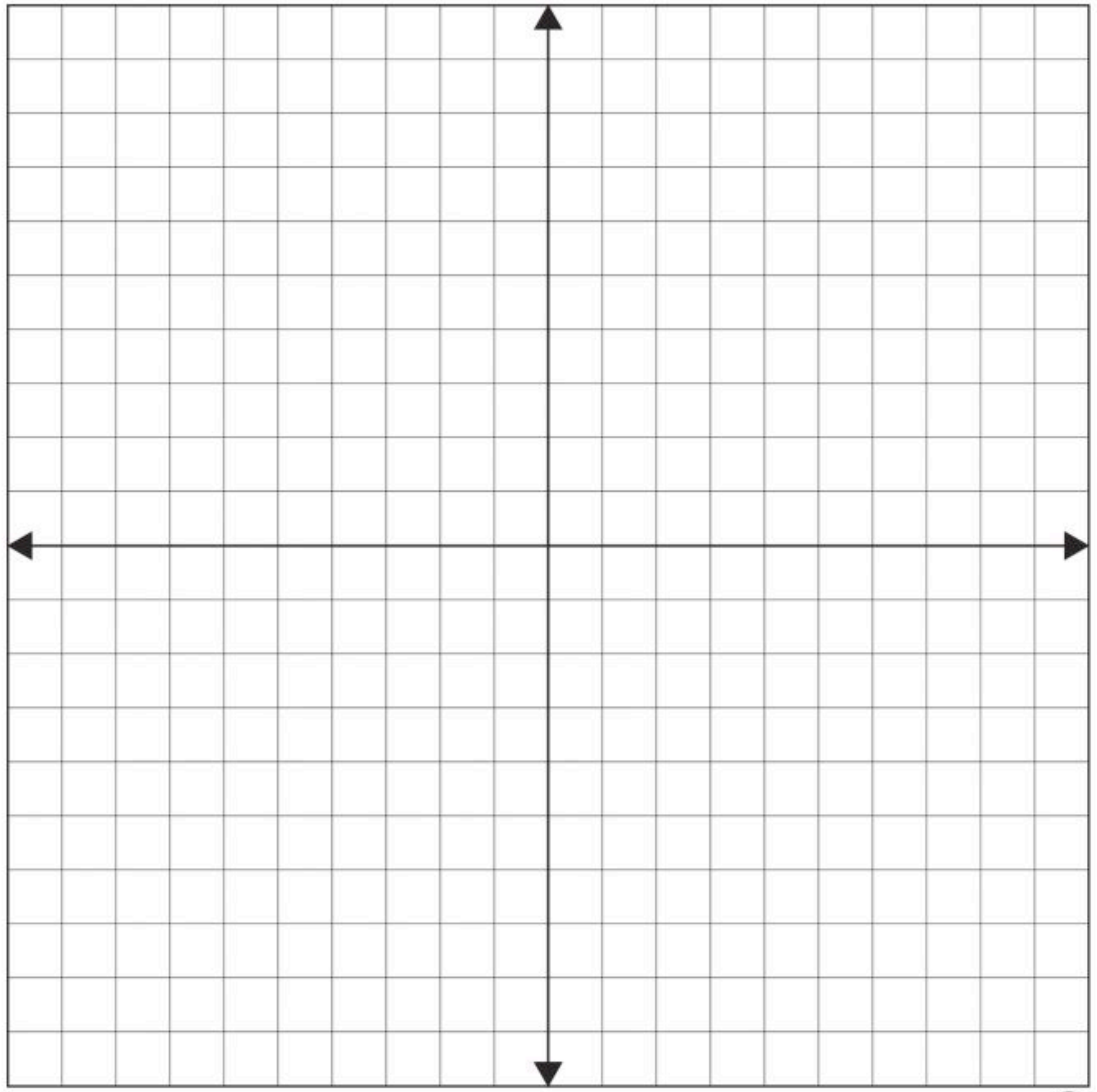

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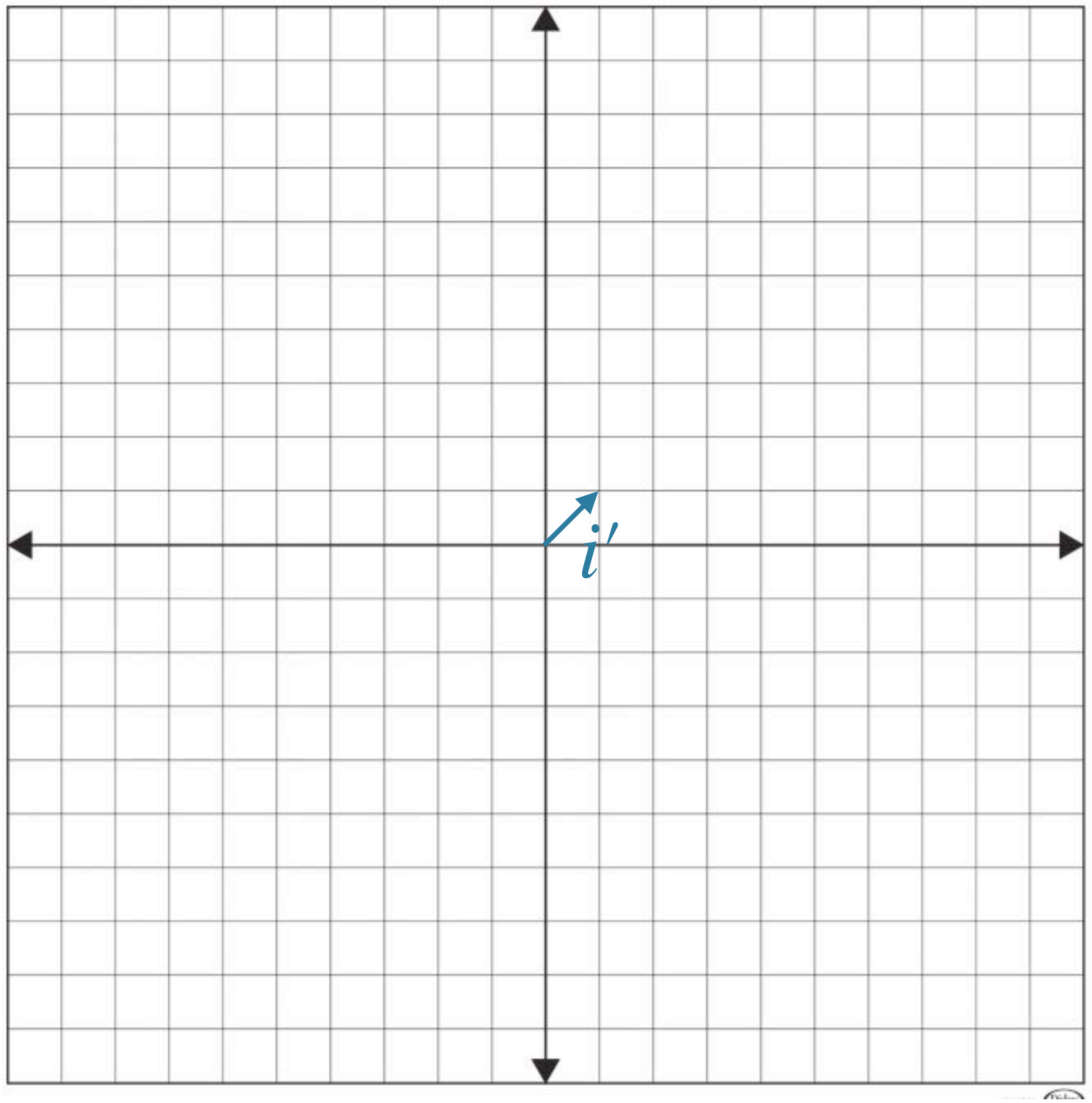

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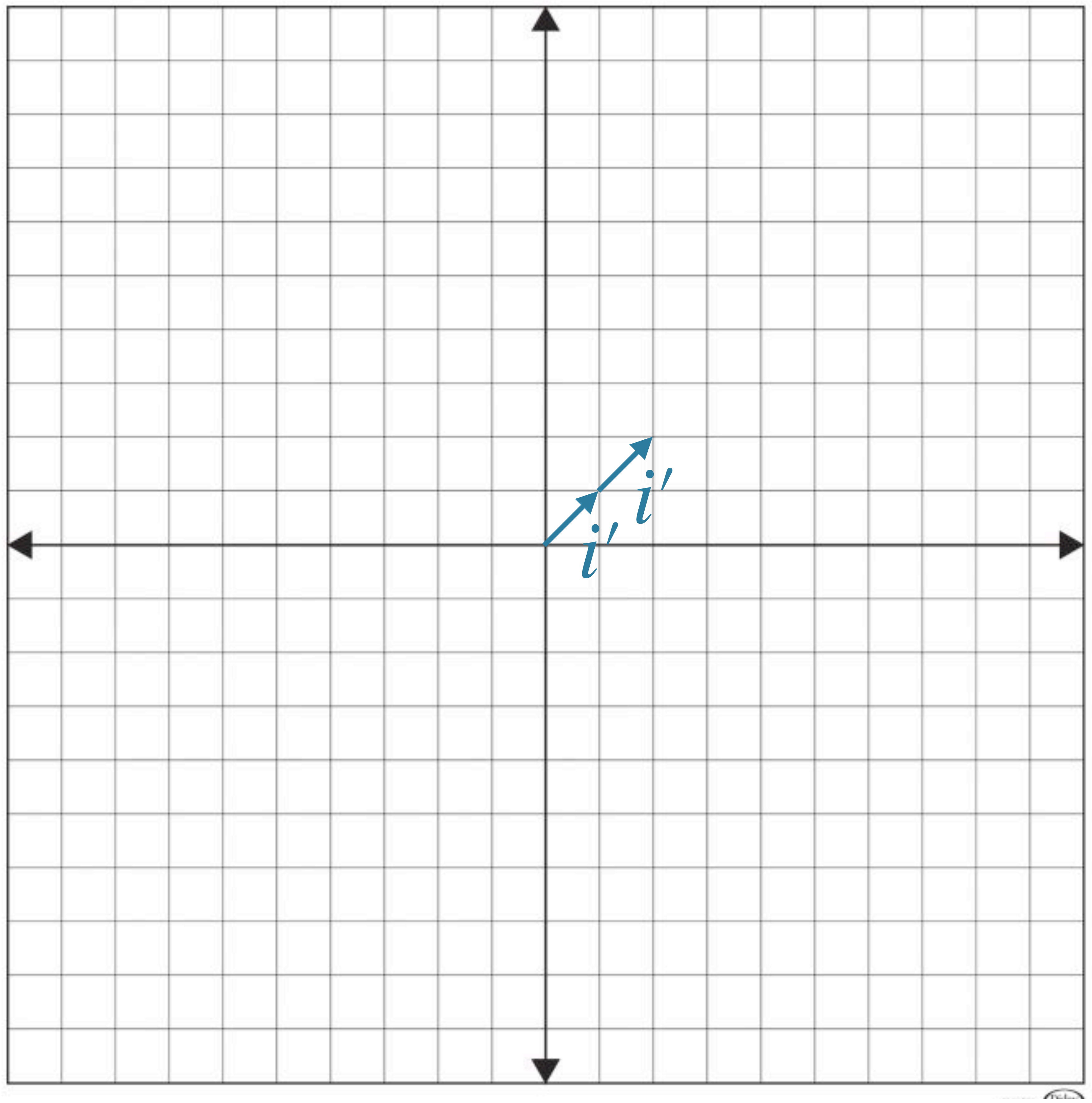

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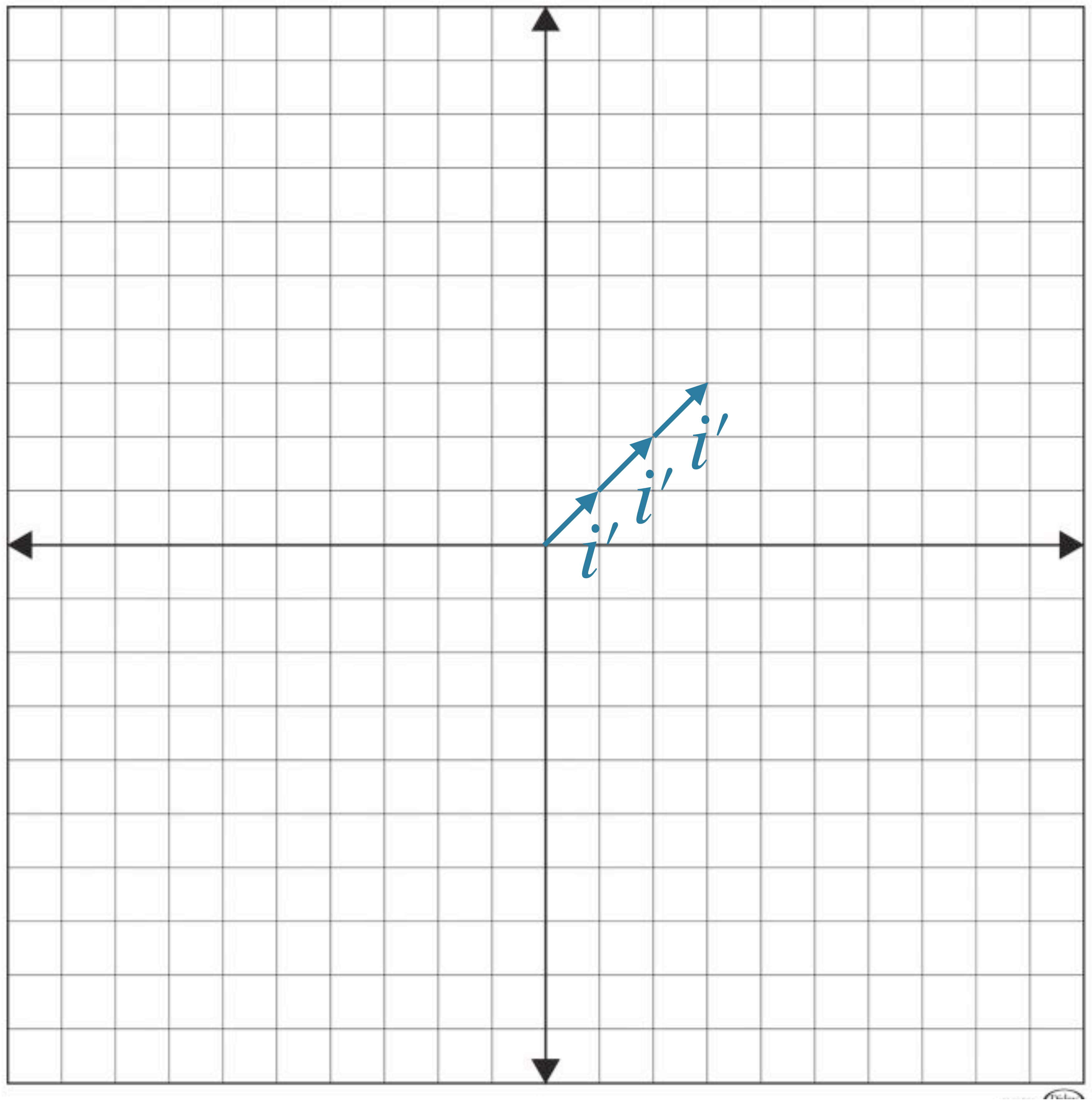

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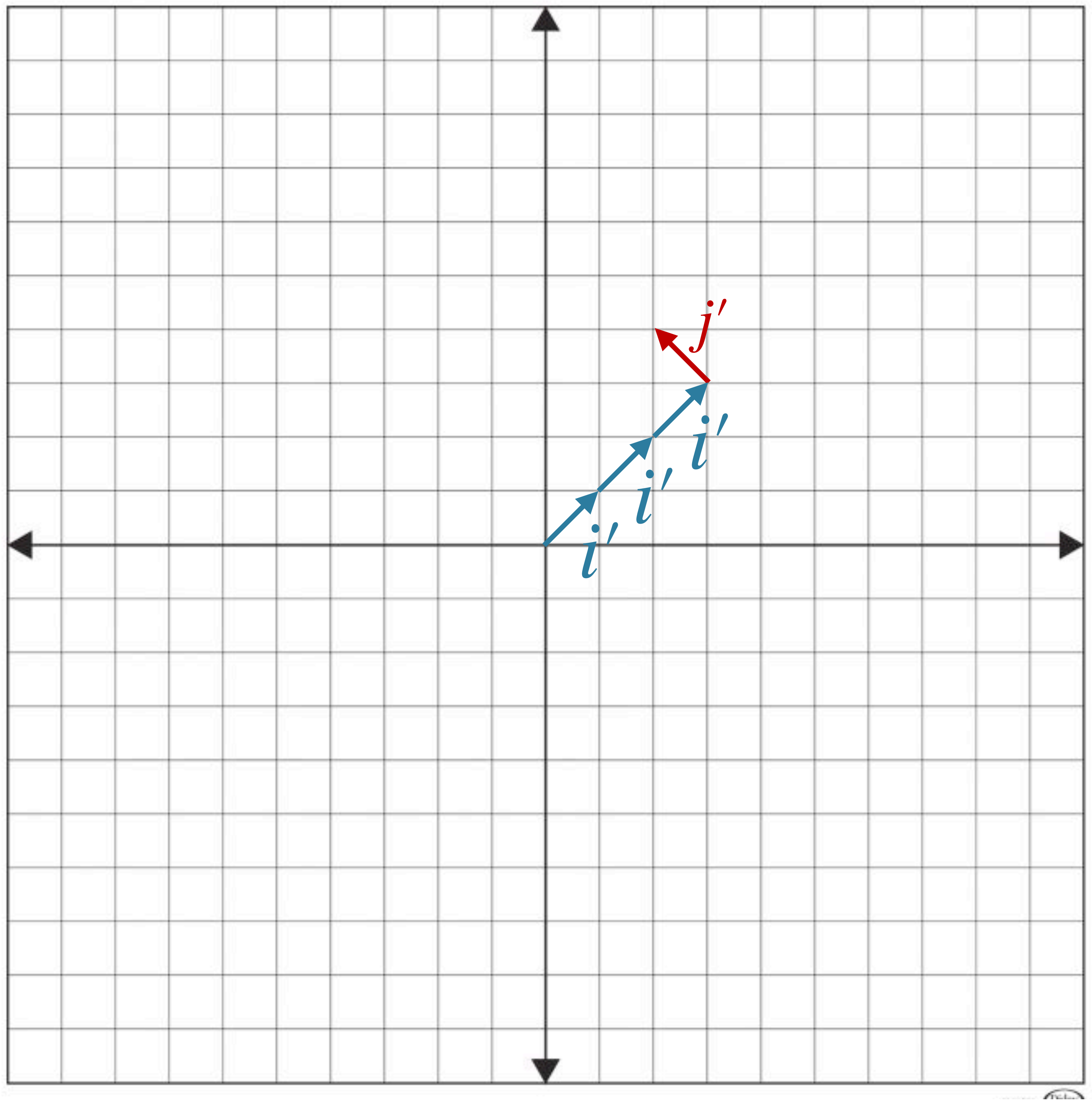

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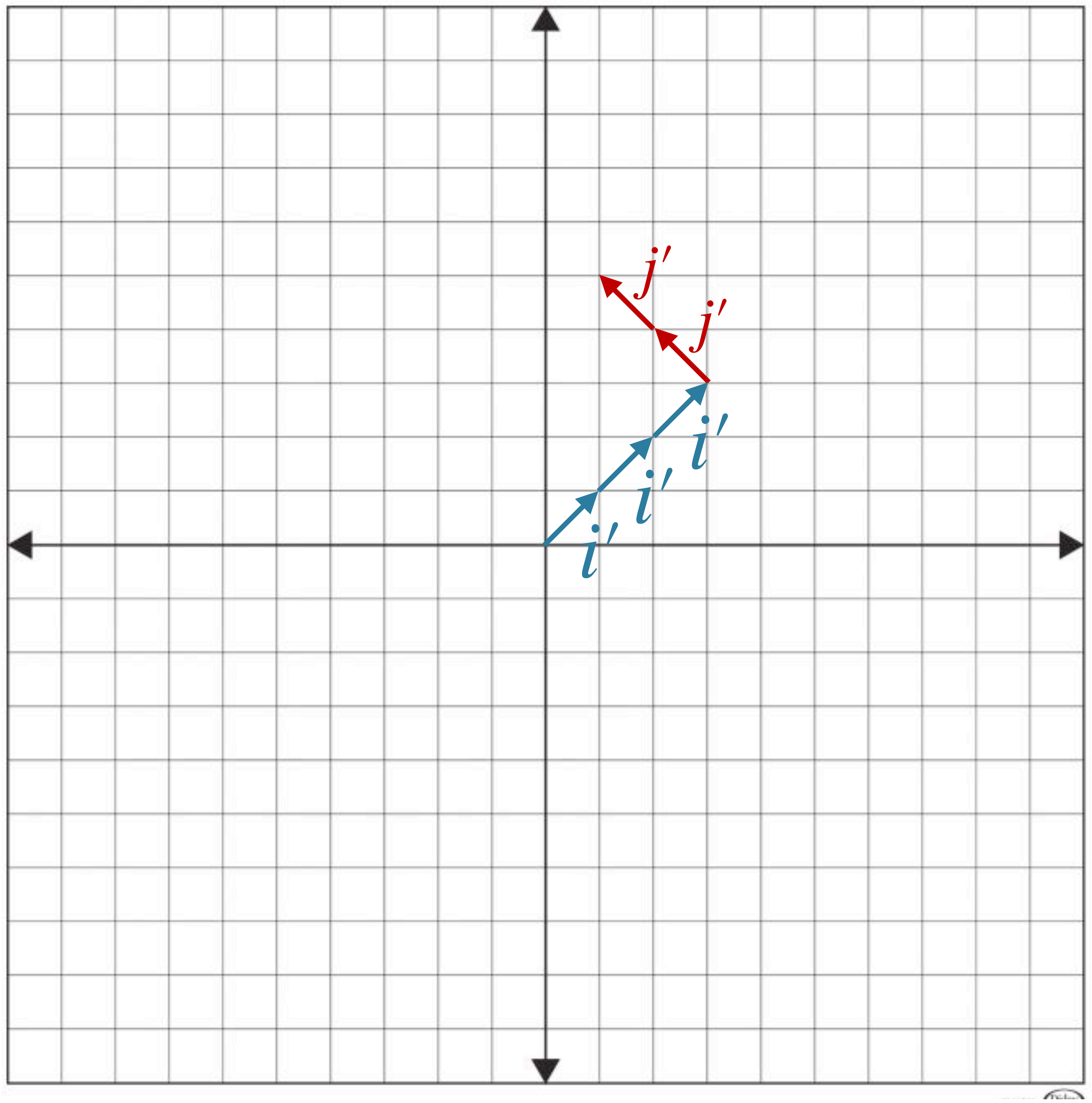

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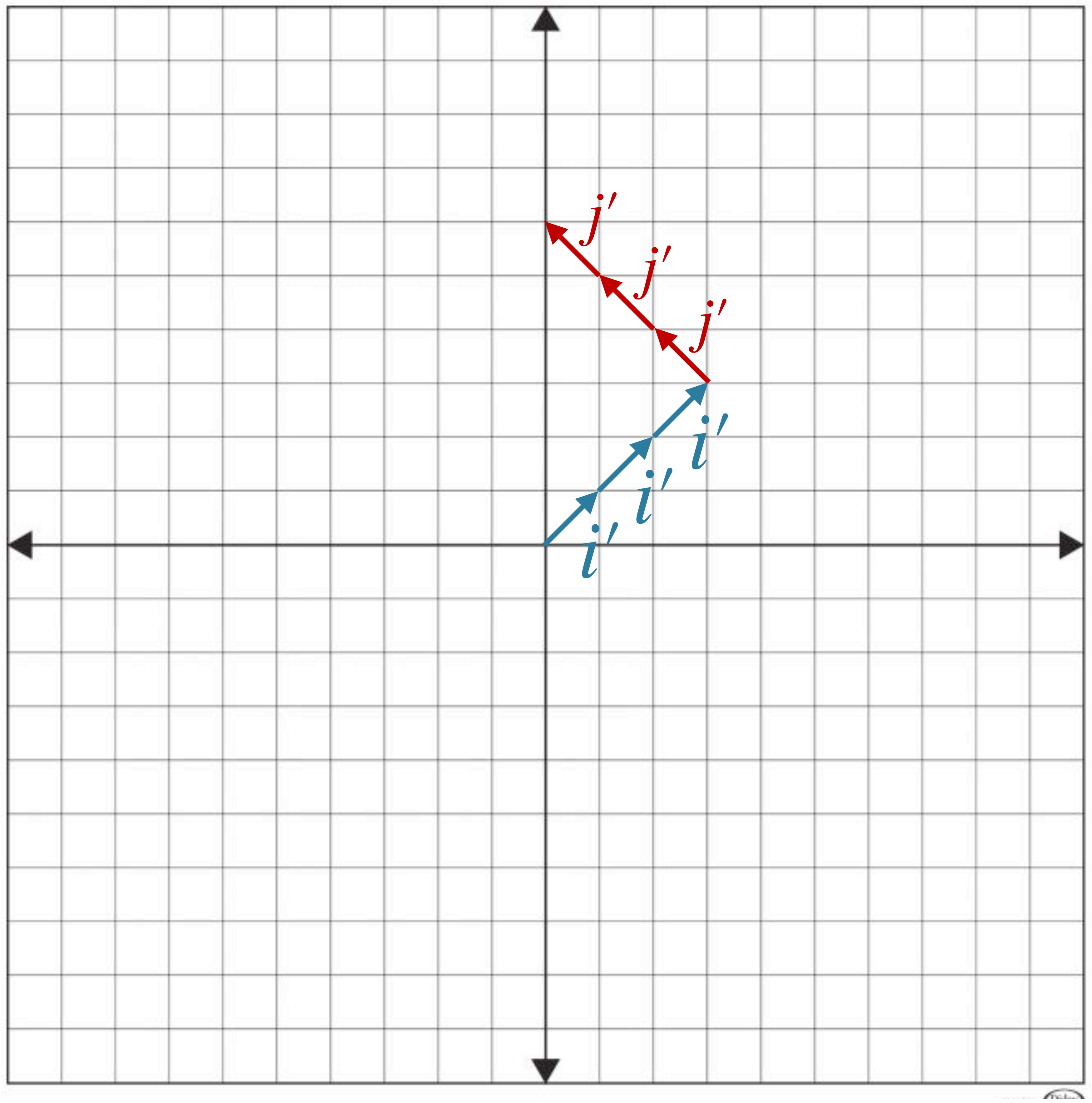

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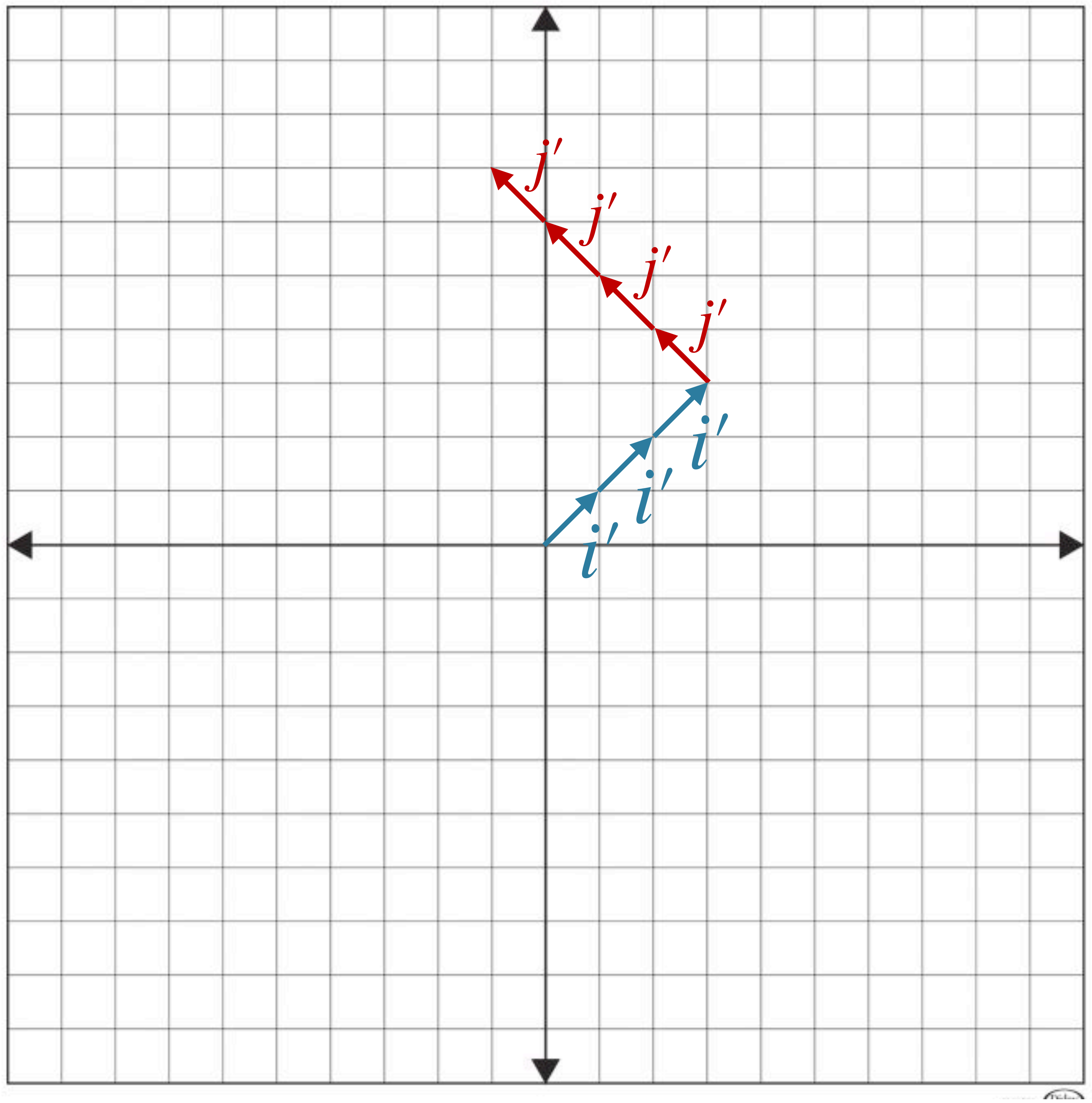

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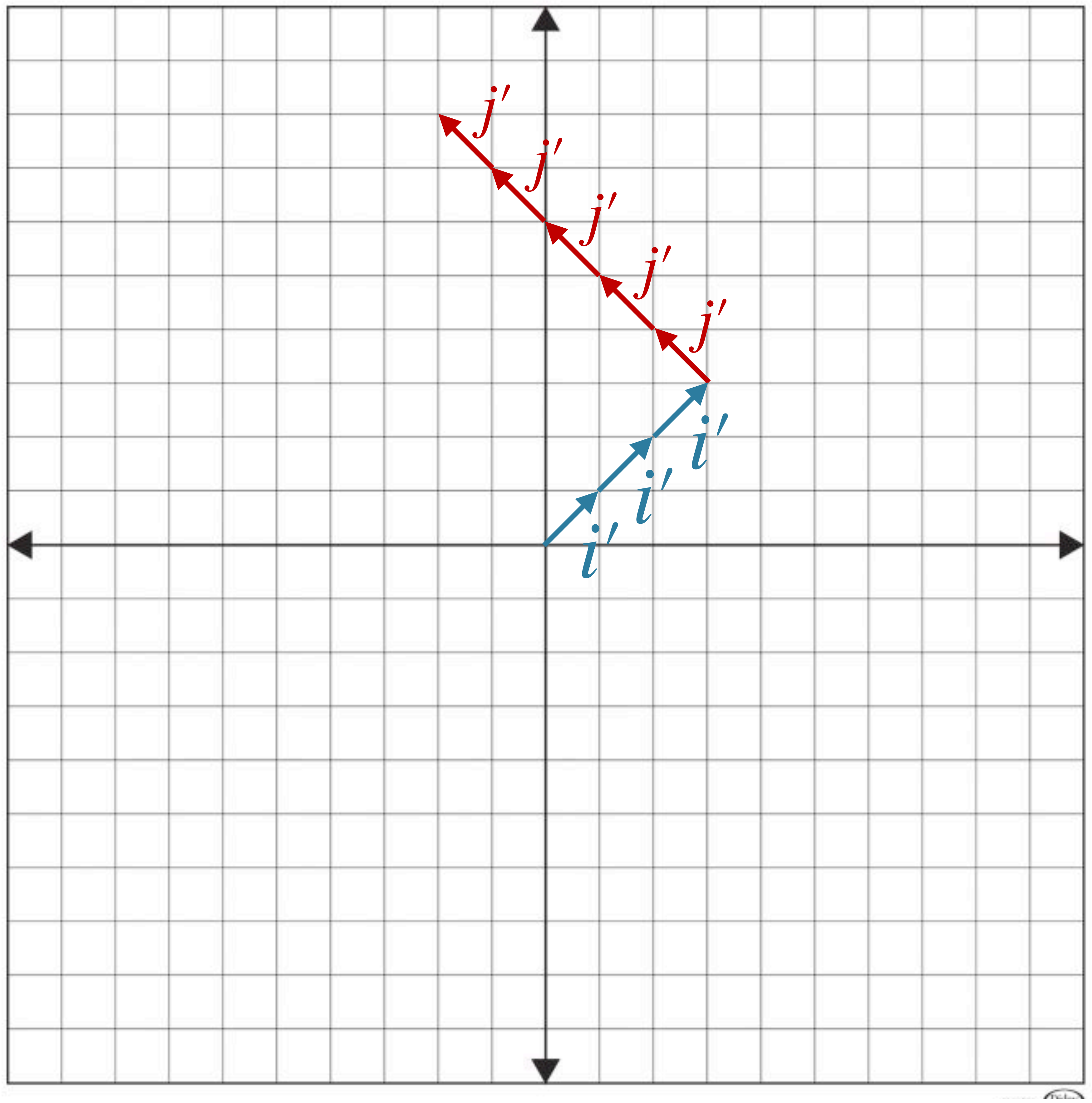

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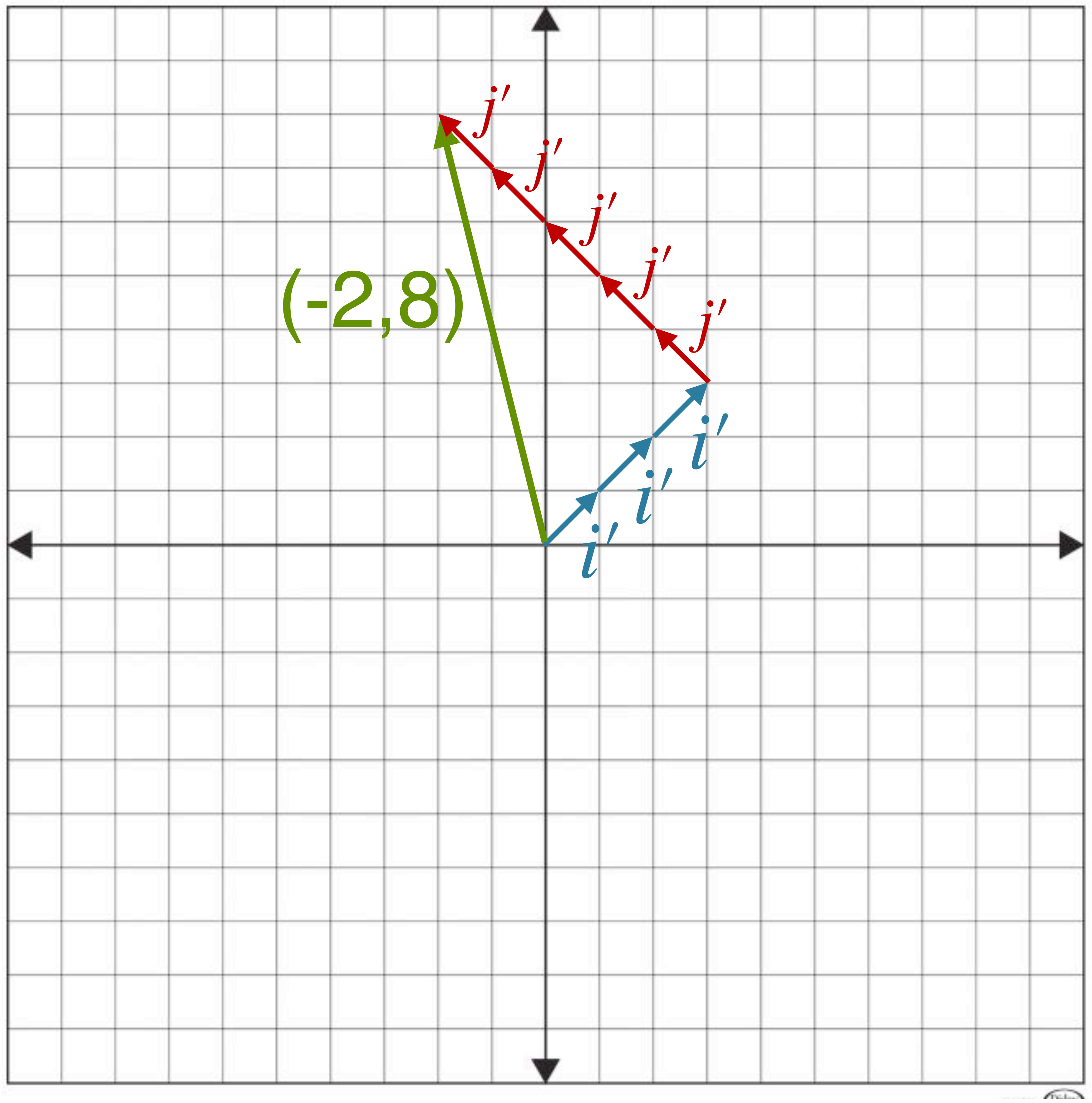

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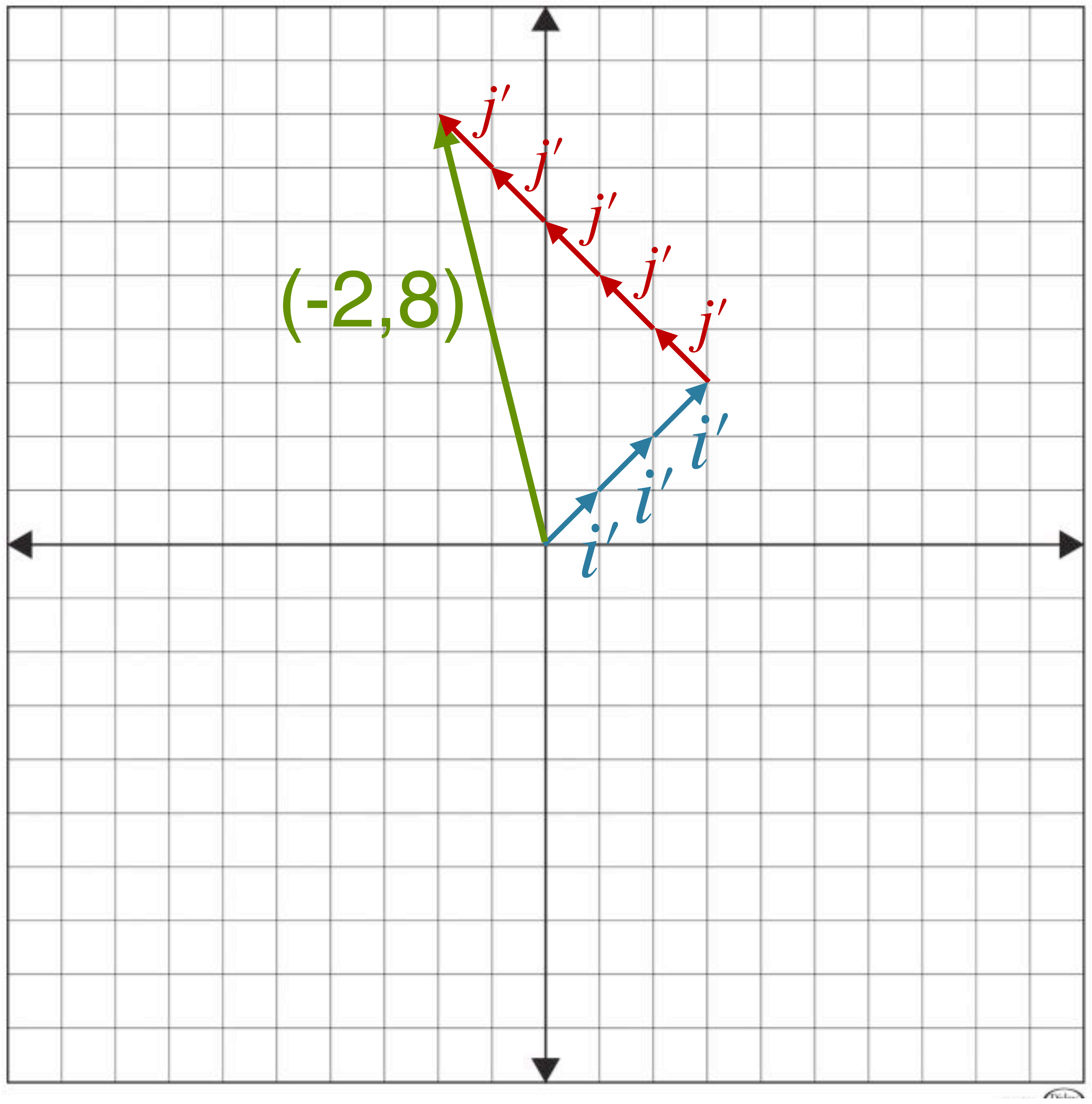

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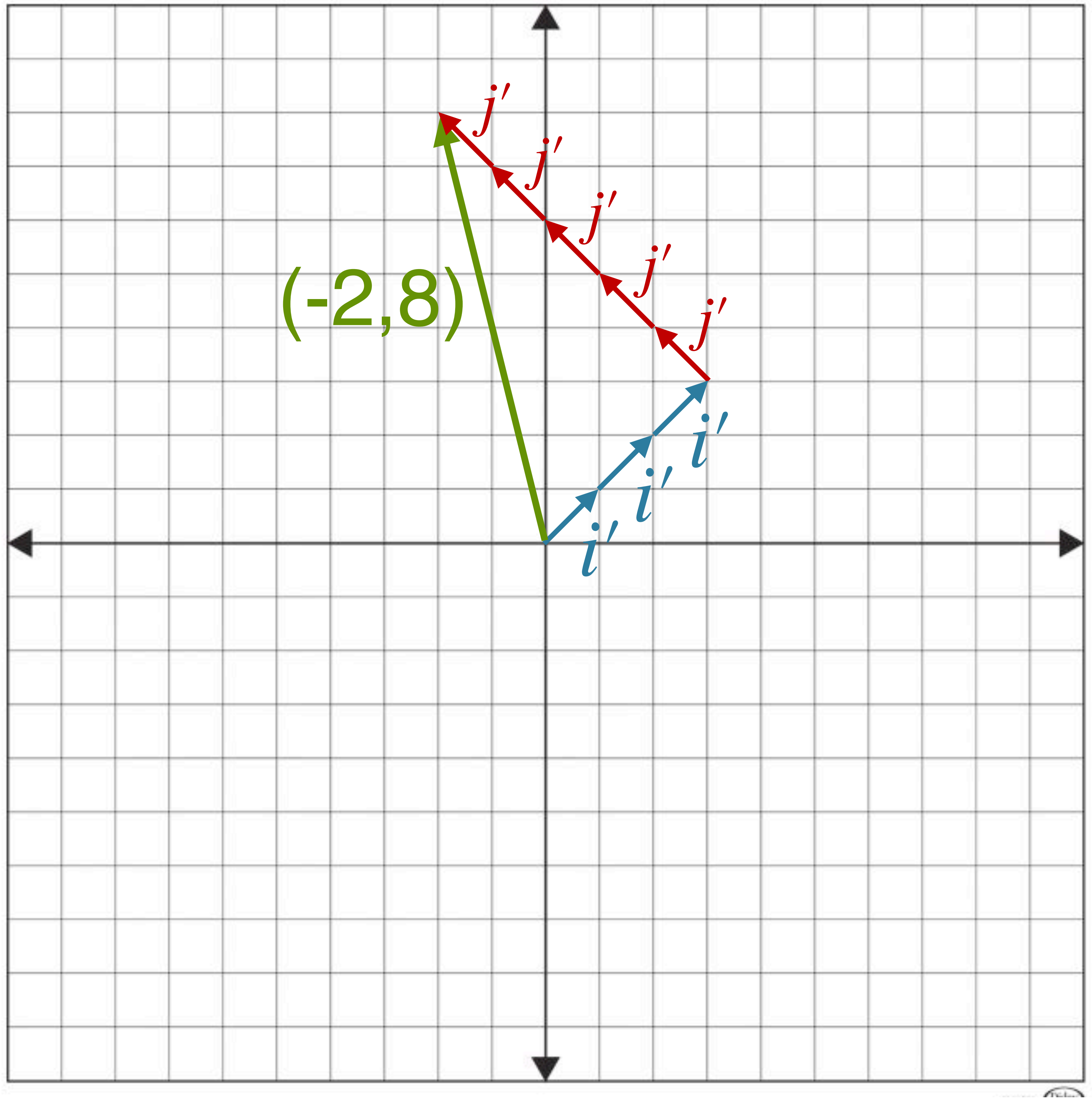

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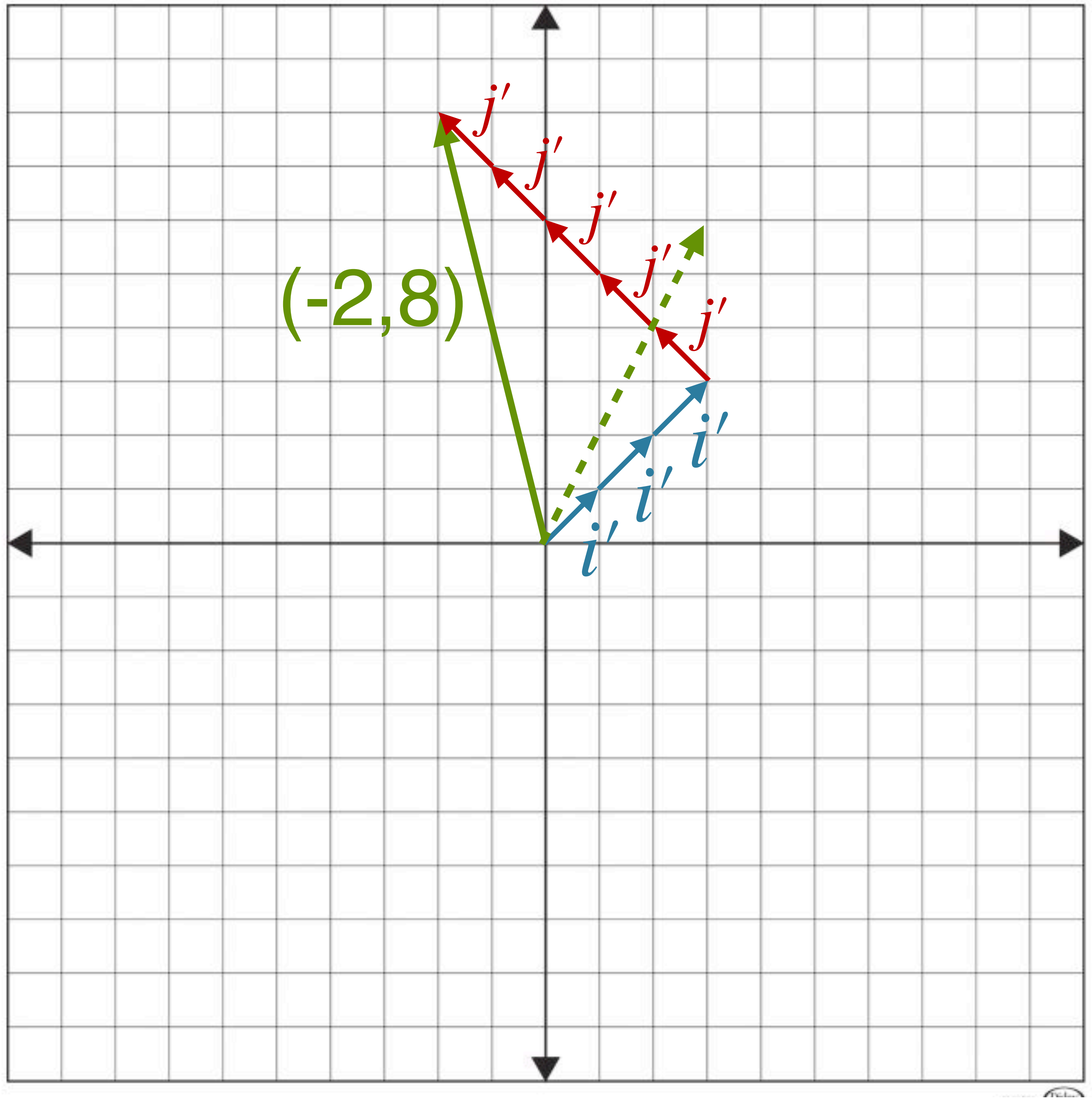

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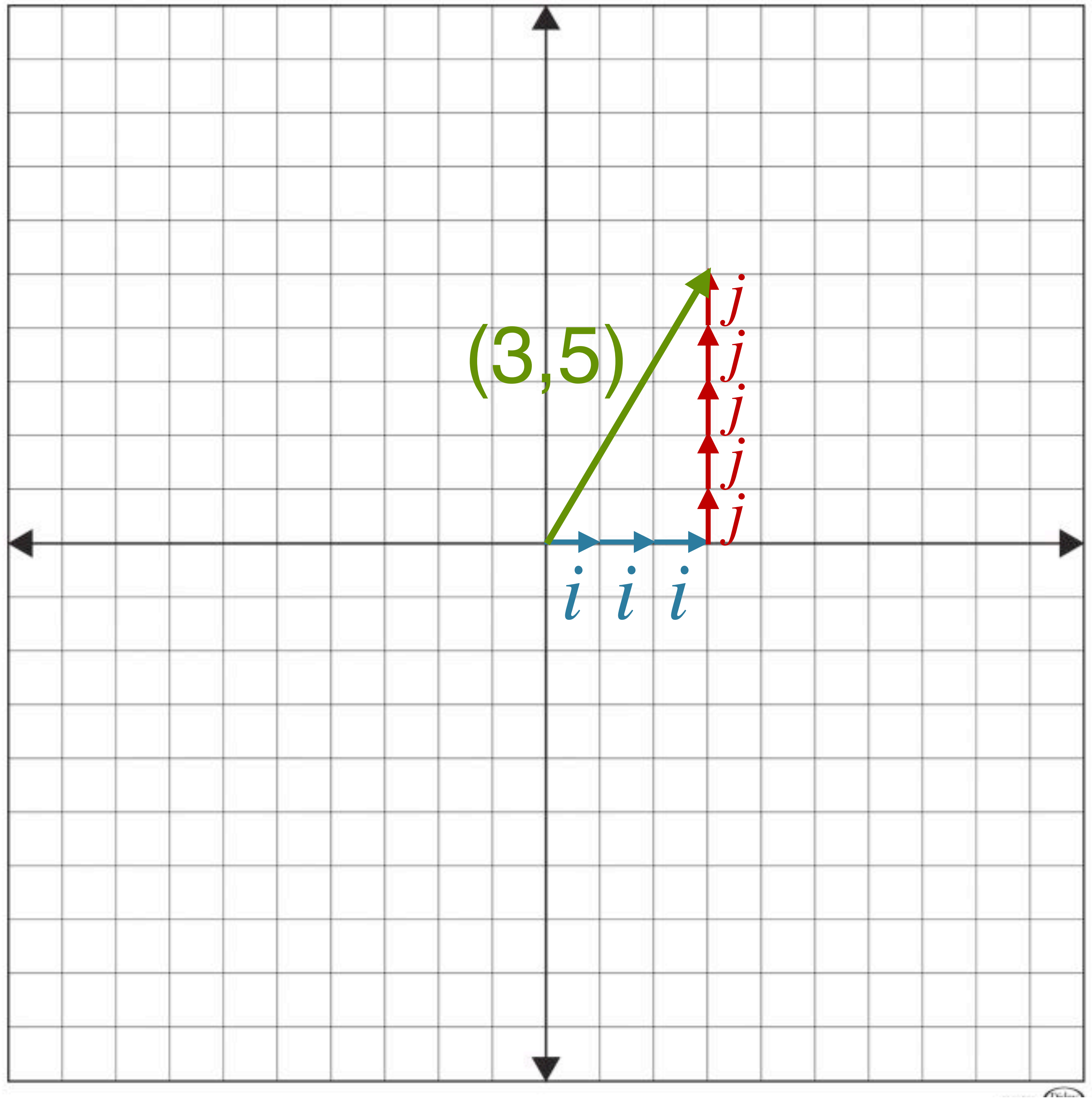
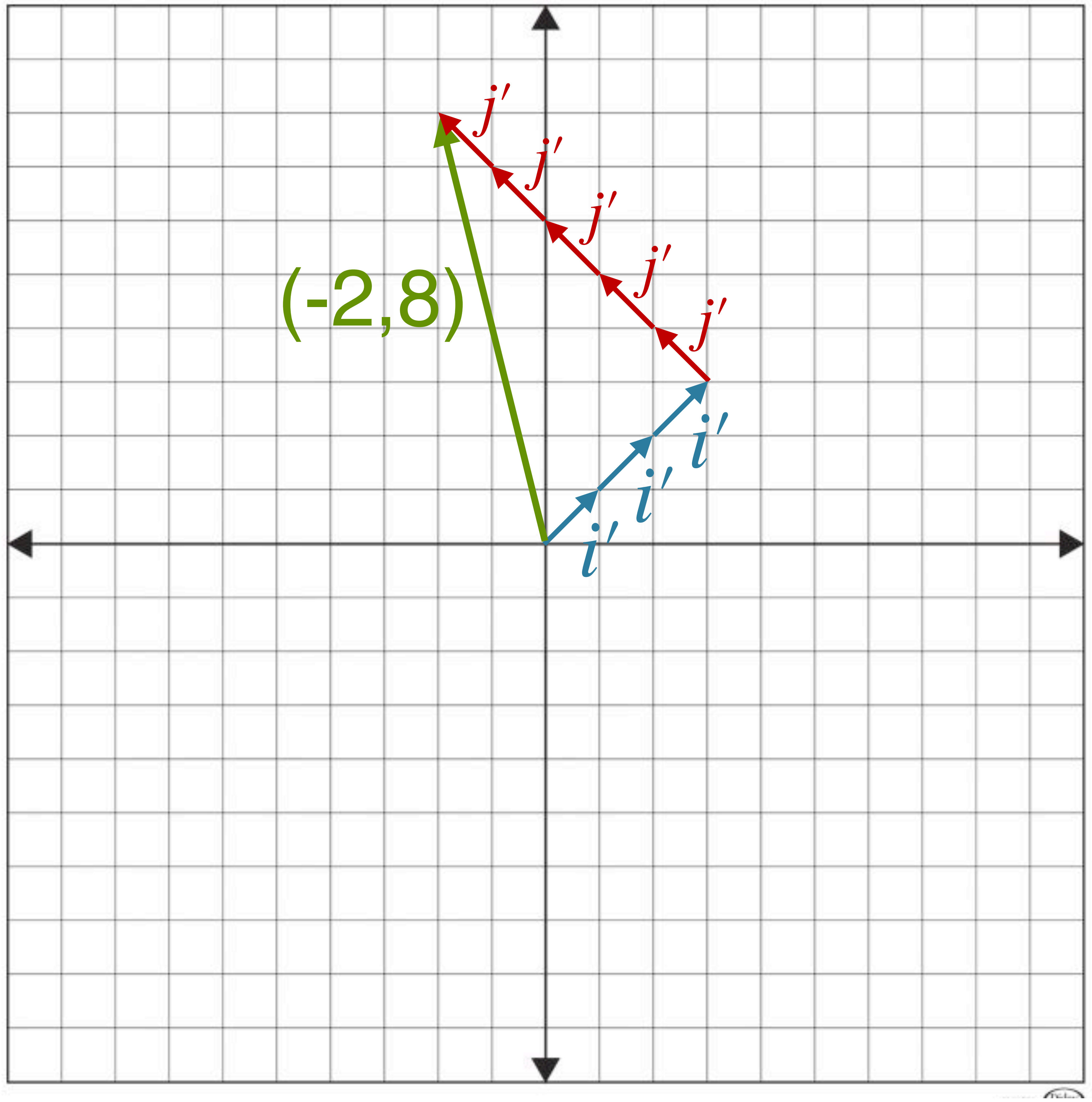
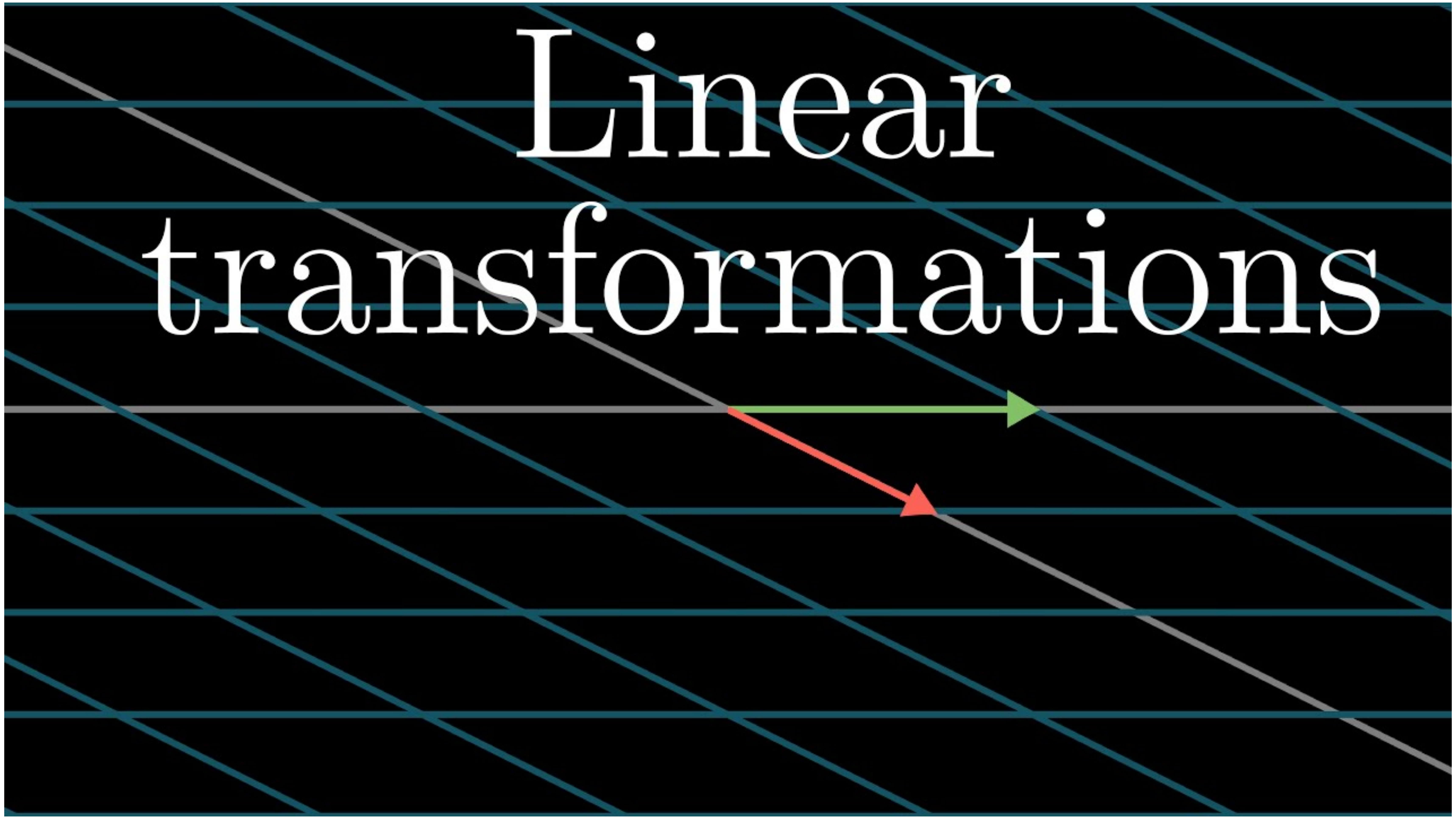




Visualizing Linear Transformations



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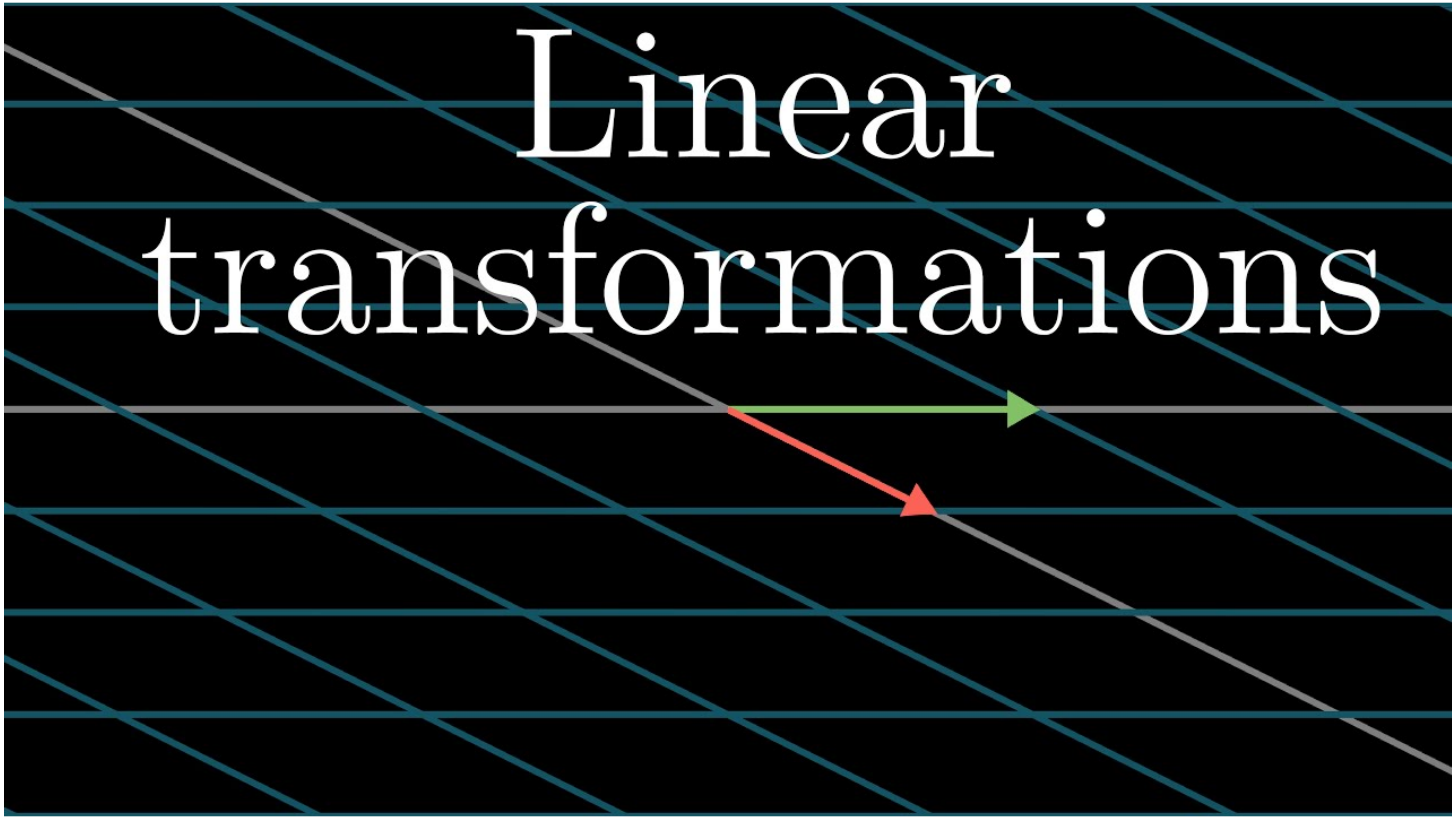

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- TLDR: Neural Nets transform vectors and vector spaces

Quiz next session!

- You should be able to:
  - **Draw vectors** on a grid
  - **Add and subtract** vectors
  - Identify when two vectors are **independent**
  - Take the **dot product** of two vectors
  - **Multiply** a vector by a matrix
  - **Multiply** two matrices
  - Give the form and discuss properties of the **Identity Matrix**
- Bring a **pen or pencil!** I will provide **calculators**