Linear Algebra

Ling 575j: Deep Learning for NLP
C.M. Downey
Spring 2023

Today's Plan

- Review vector and matrix operations
- Discuss vector independence and span
- Dissect matrix multiplication
- Introduce linear transformations

Linear Algebra Objects

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- **Scalars**
  - Single numbers
  - What you're used to elsewhere in math
  - examples: 0, 1, 3.14, π, 7/22

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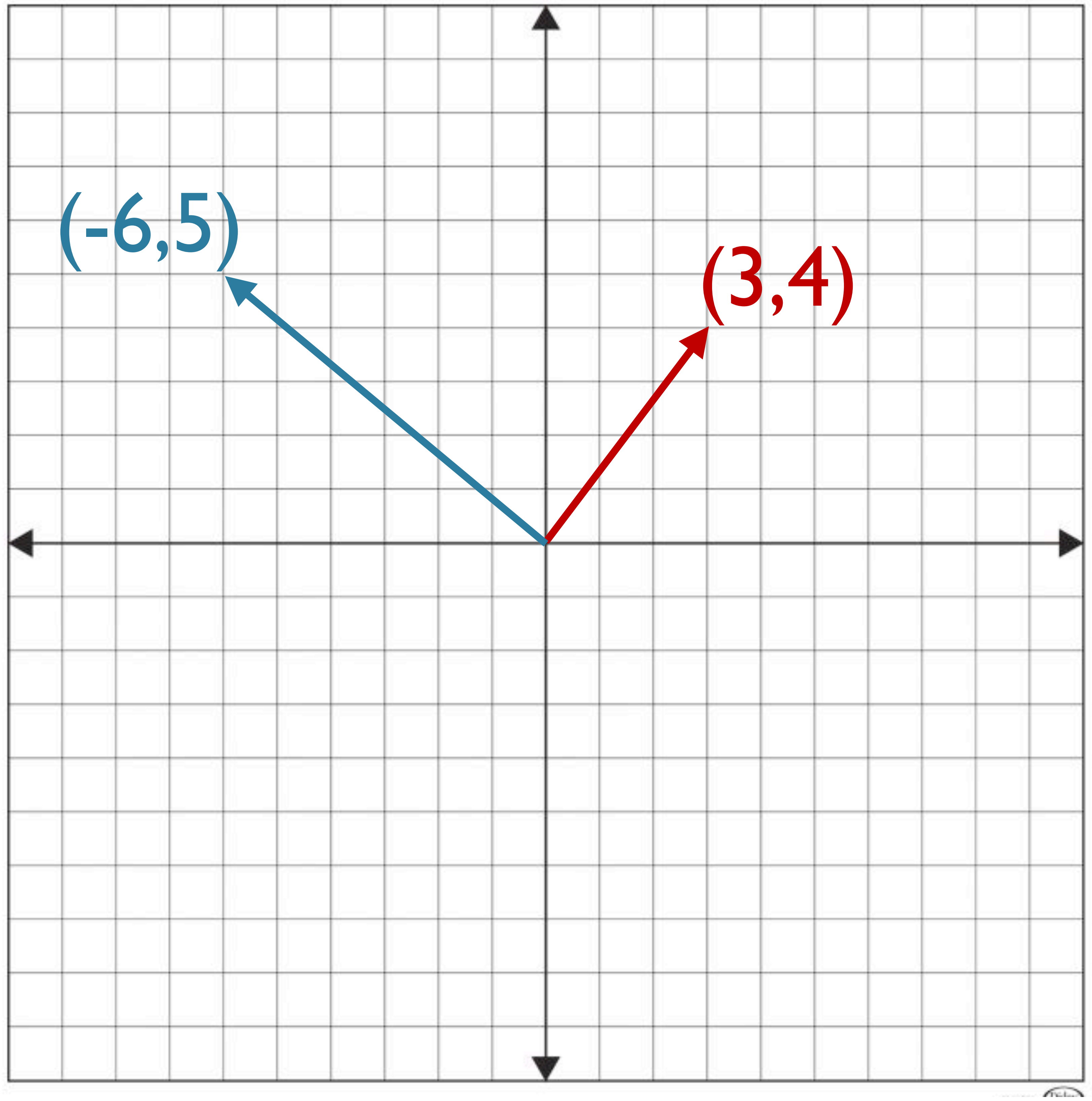

Vector Properties

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$$c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$$

(c is a scalar)

Vector Spans and Spaces

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- Definition applies to **any number** of vectors and constants
- Note: $a = \mathbf{0}$ is used to indicate a vector of zeros

Vector (In)dependence Examples

- What constants solve this equation?

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$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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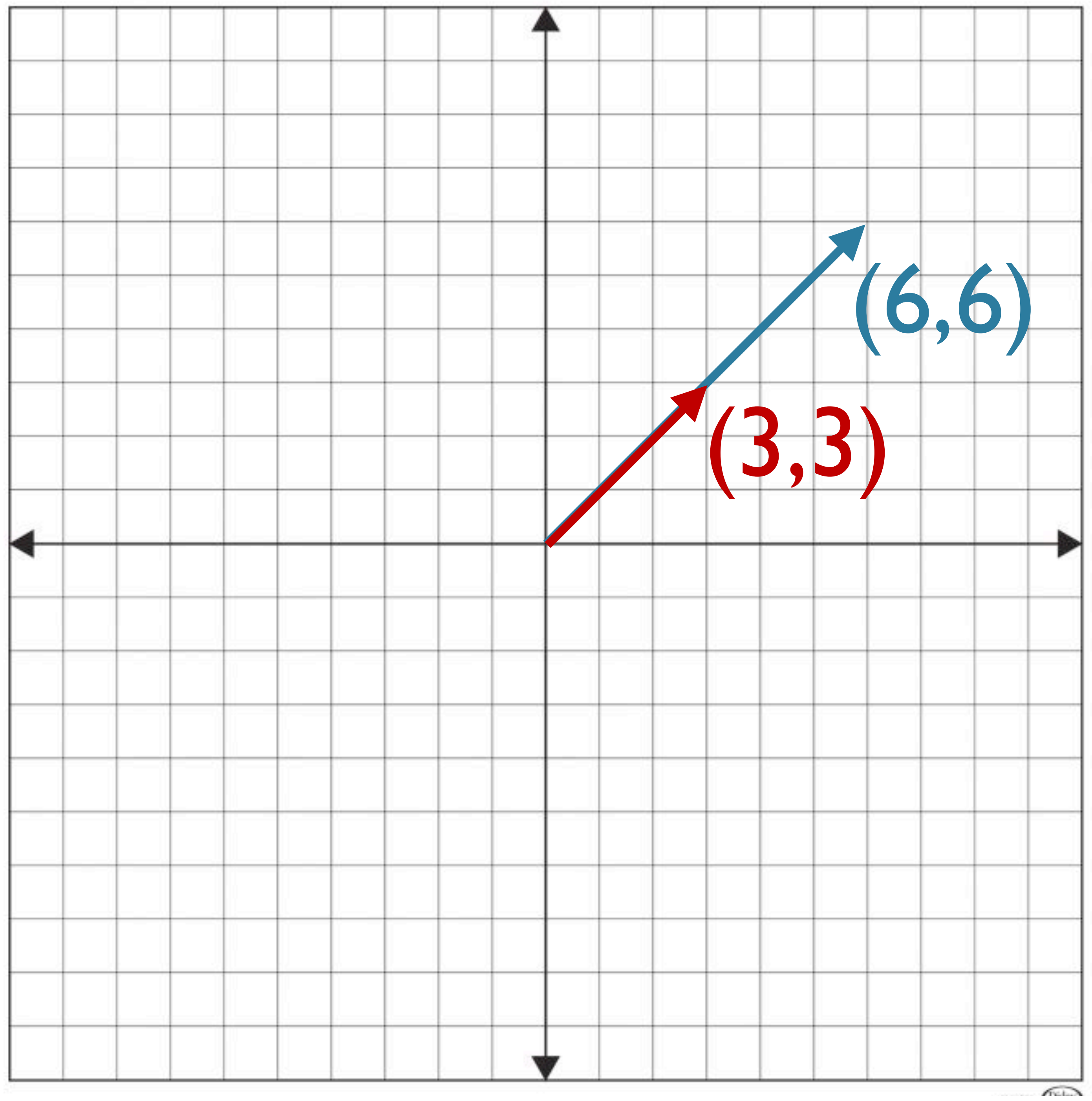

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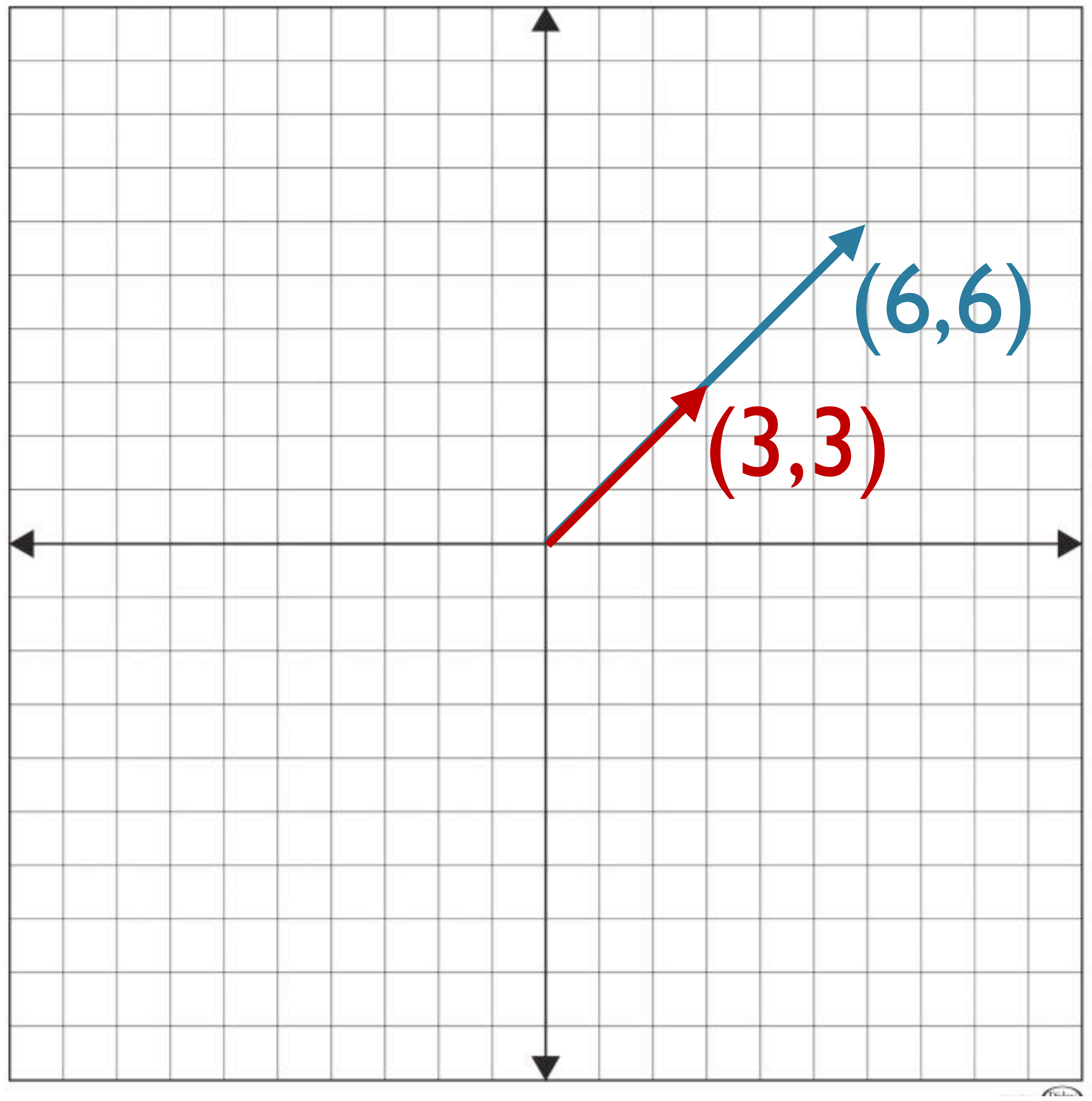

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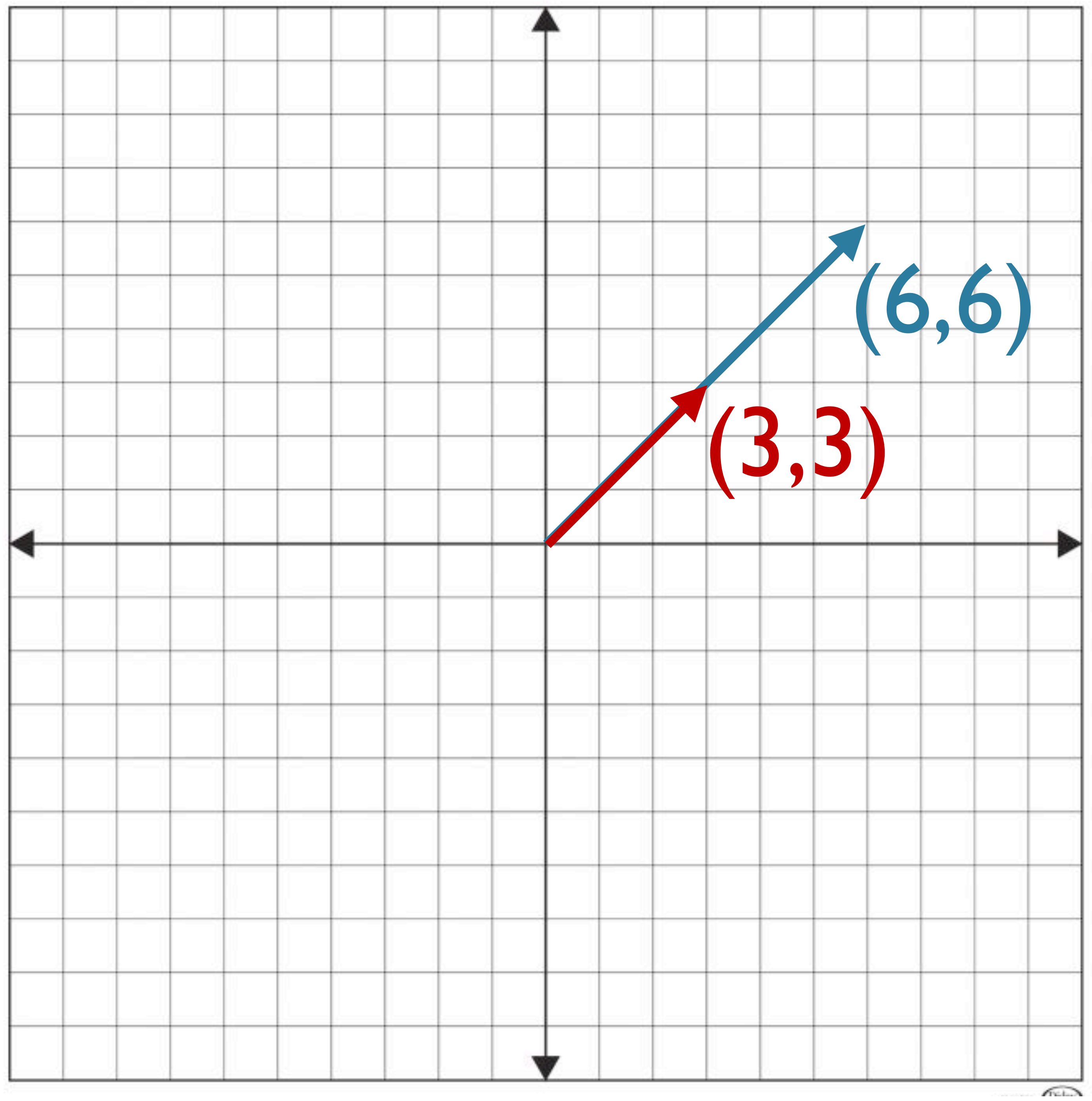

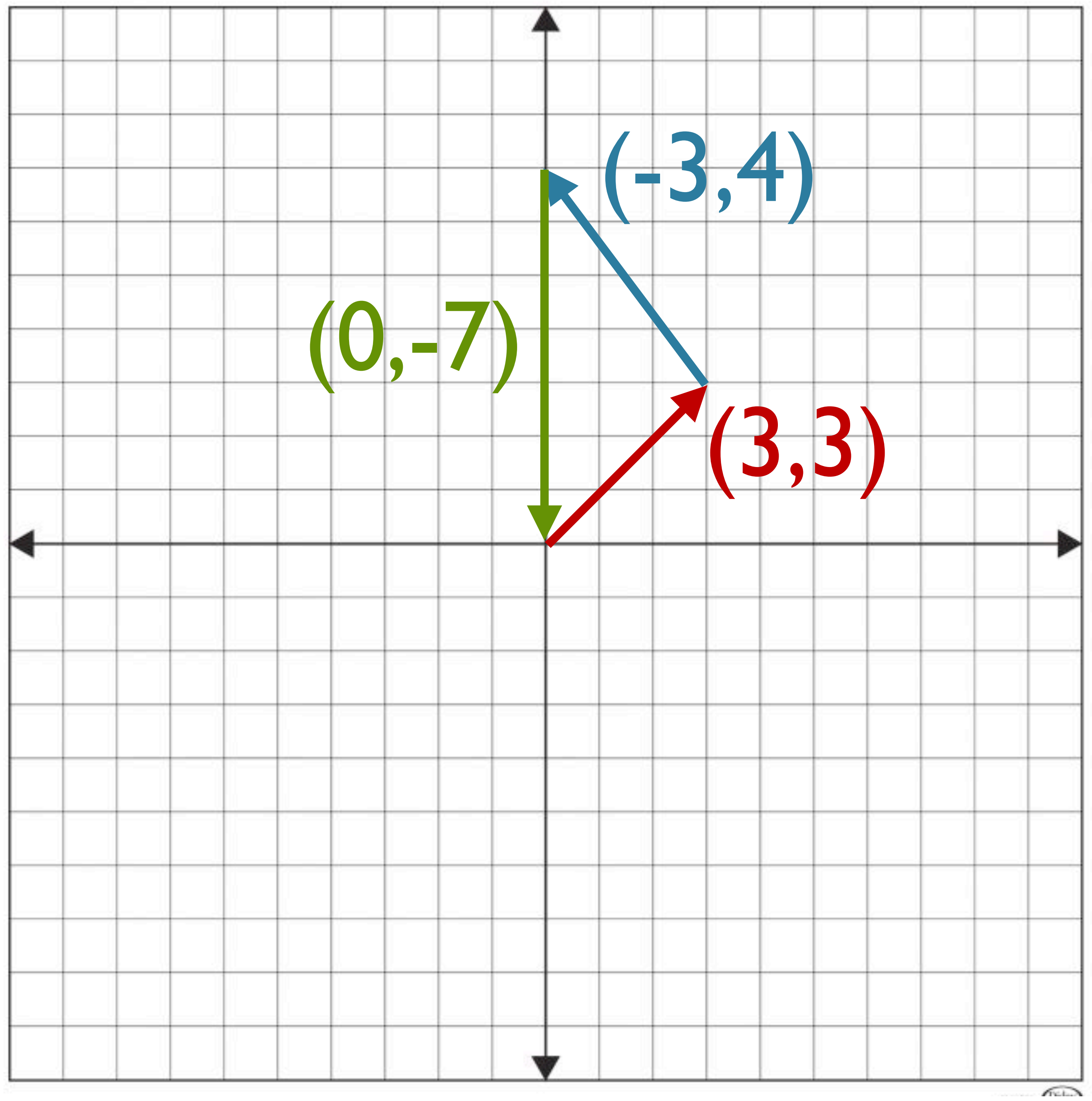

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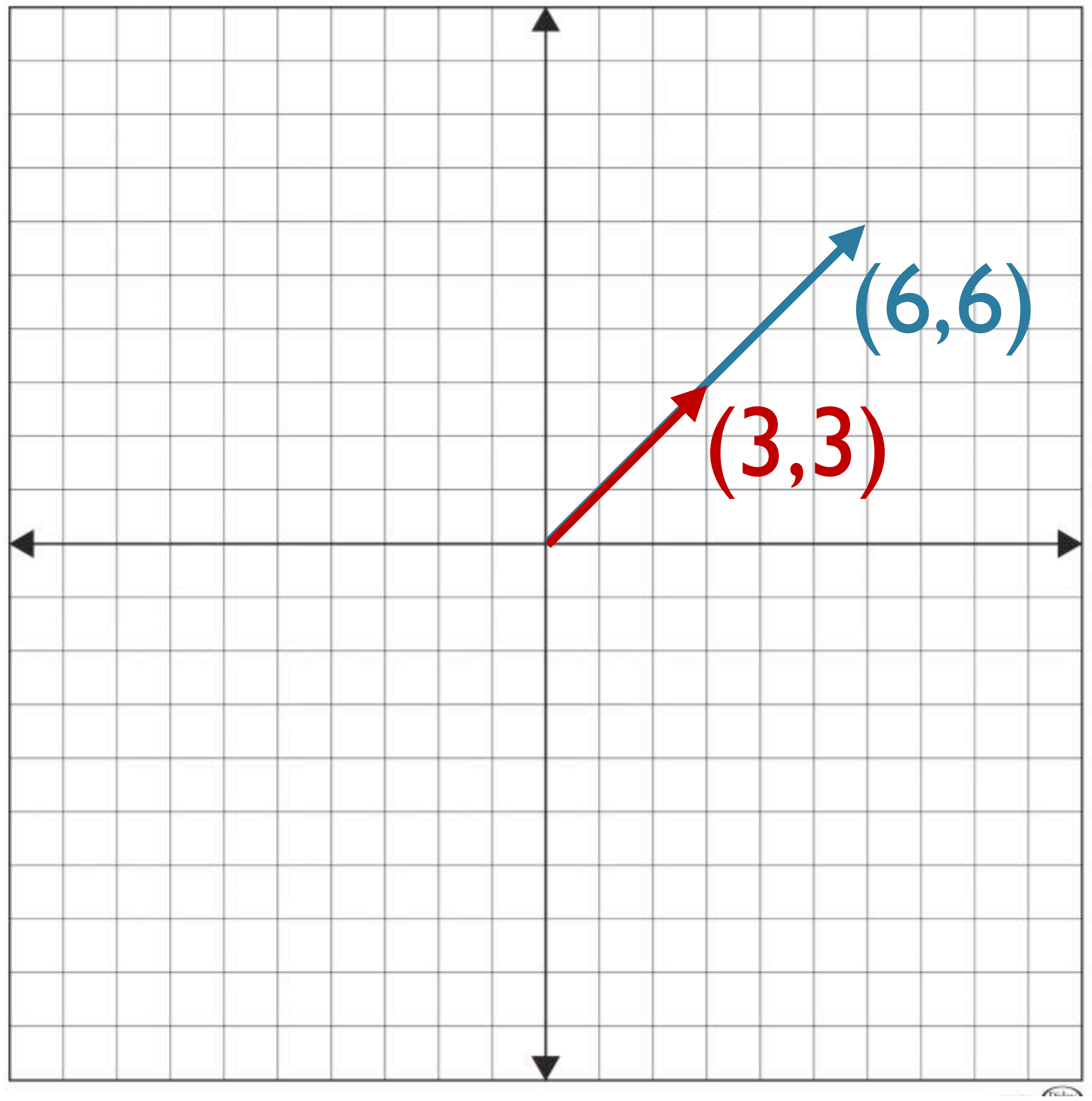

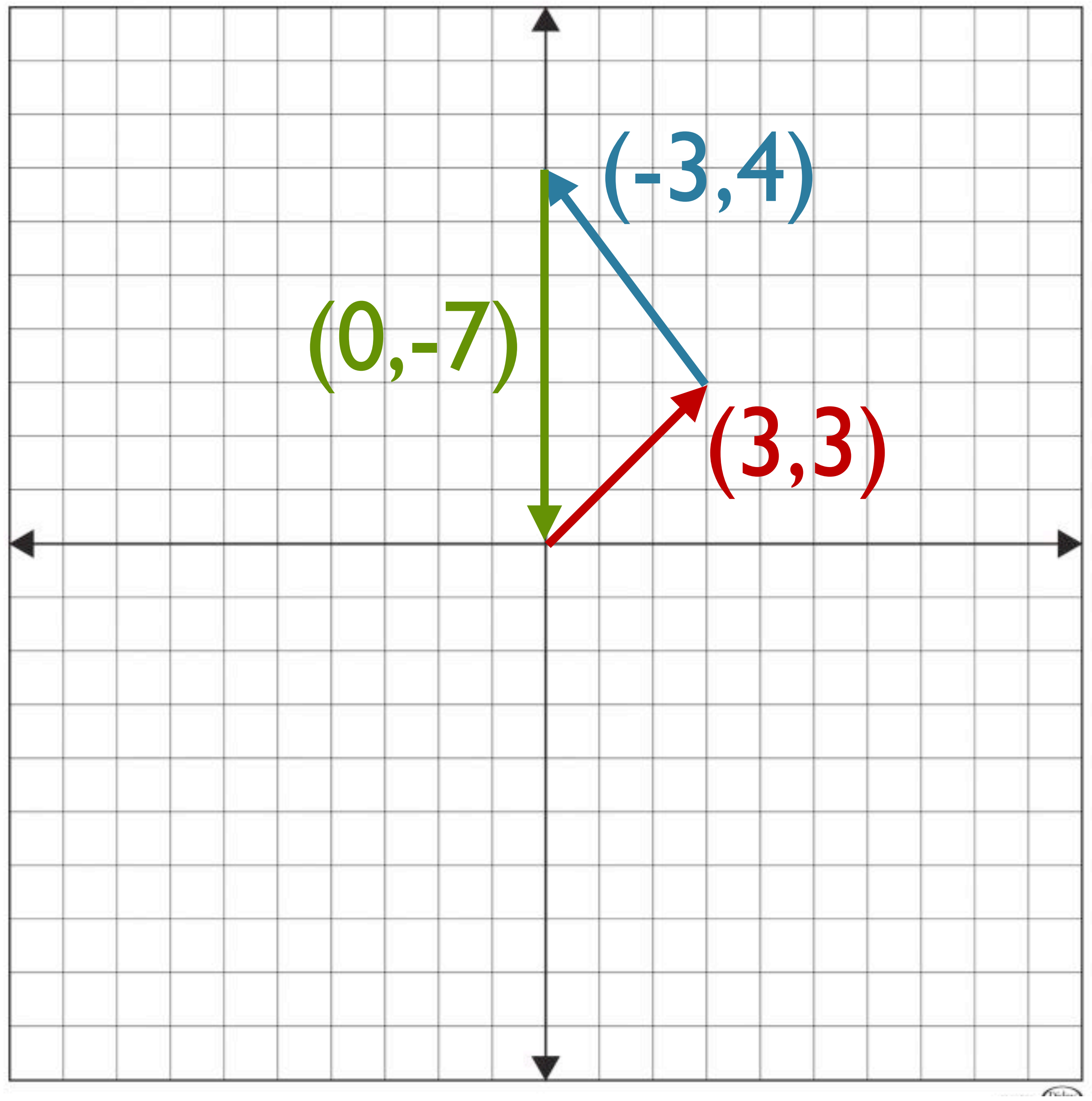

(this is what adding vectors looks like)

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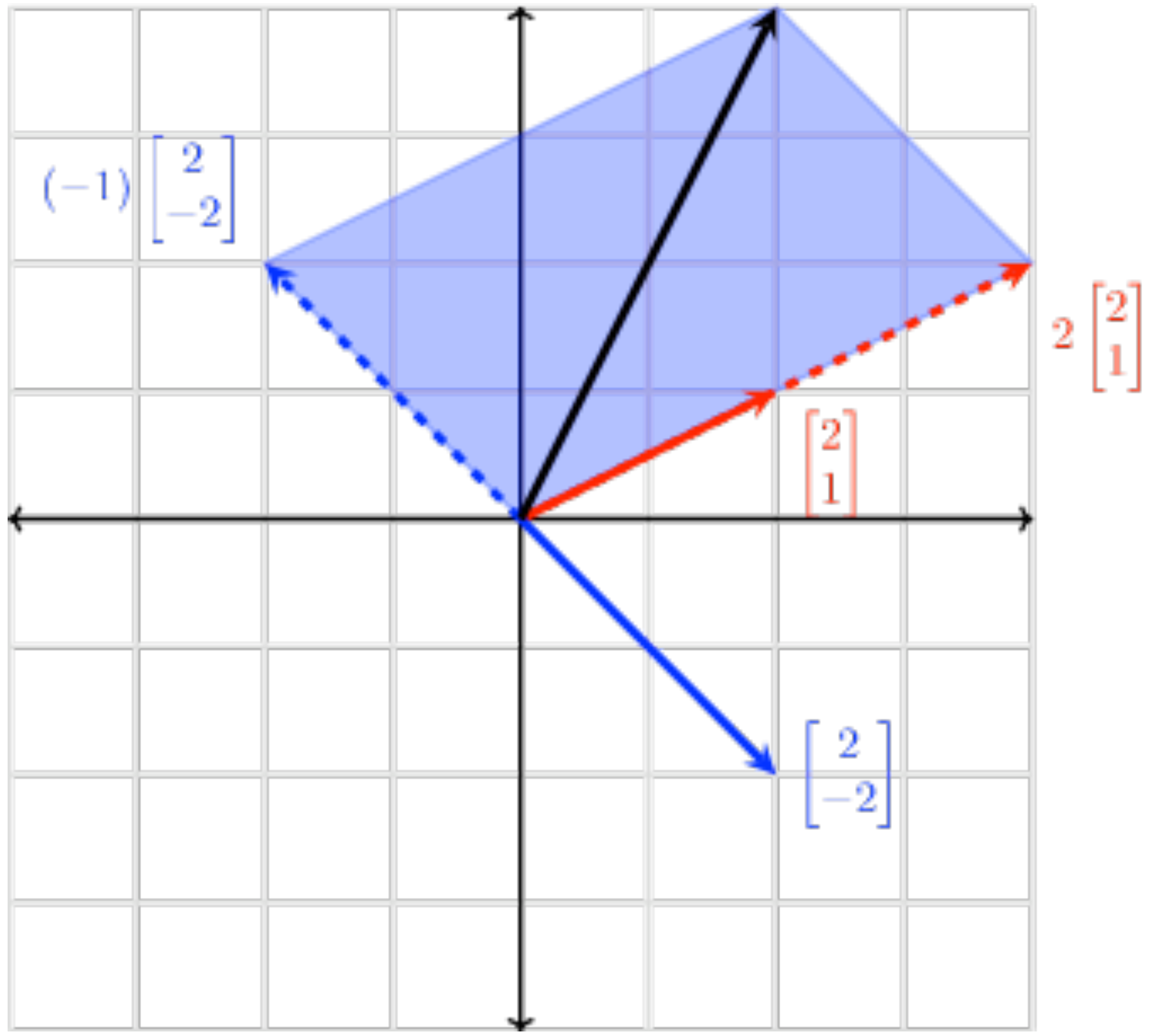

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  - Ex: **a** and **b** above span a **2-D plane in** R^3

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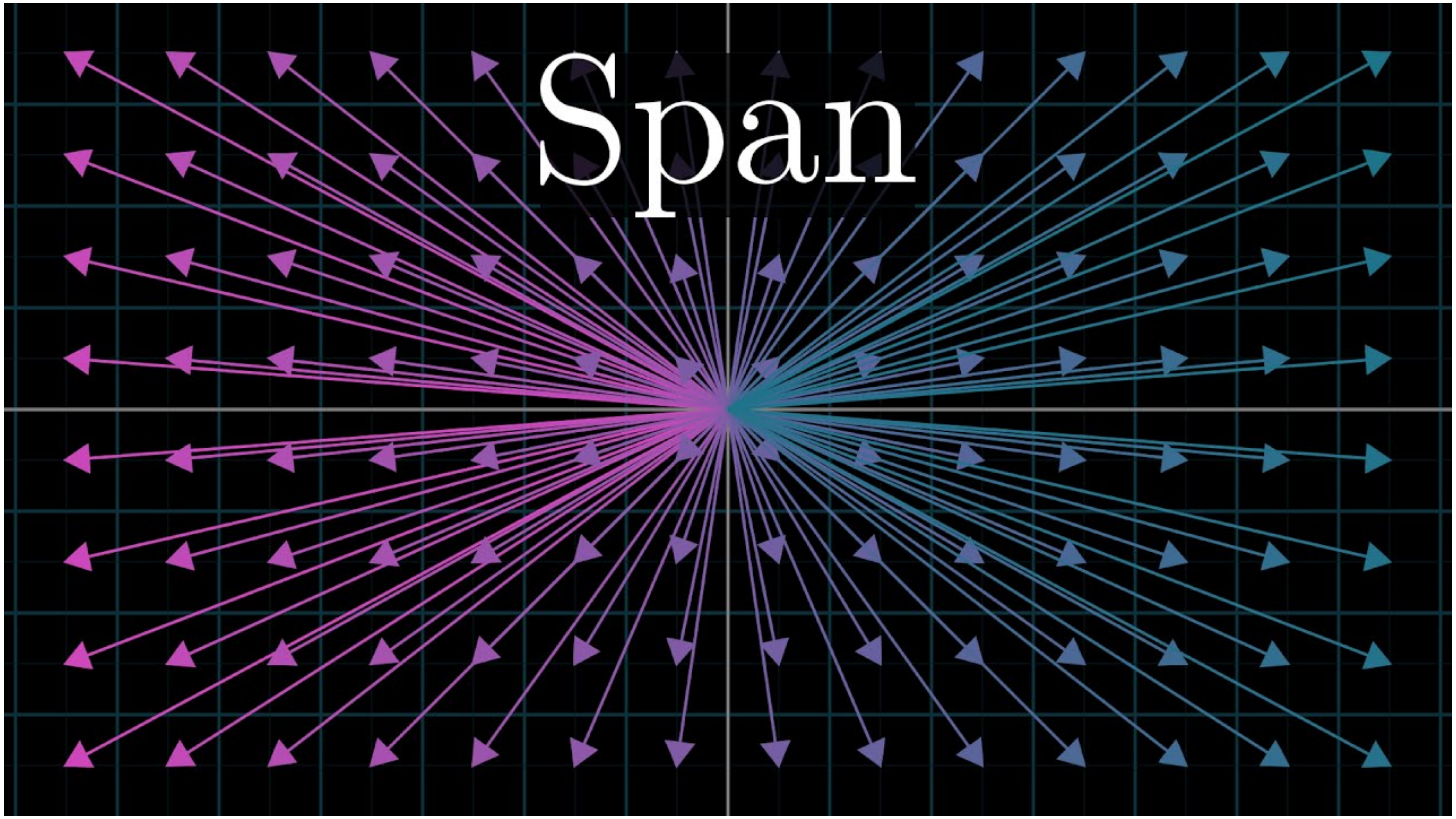
Span Video



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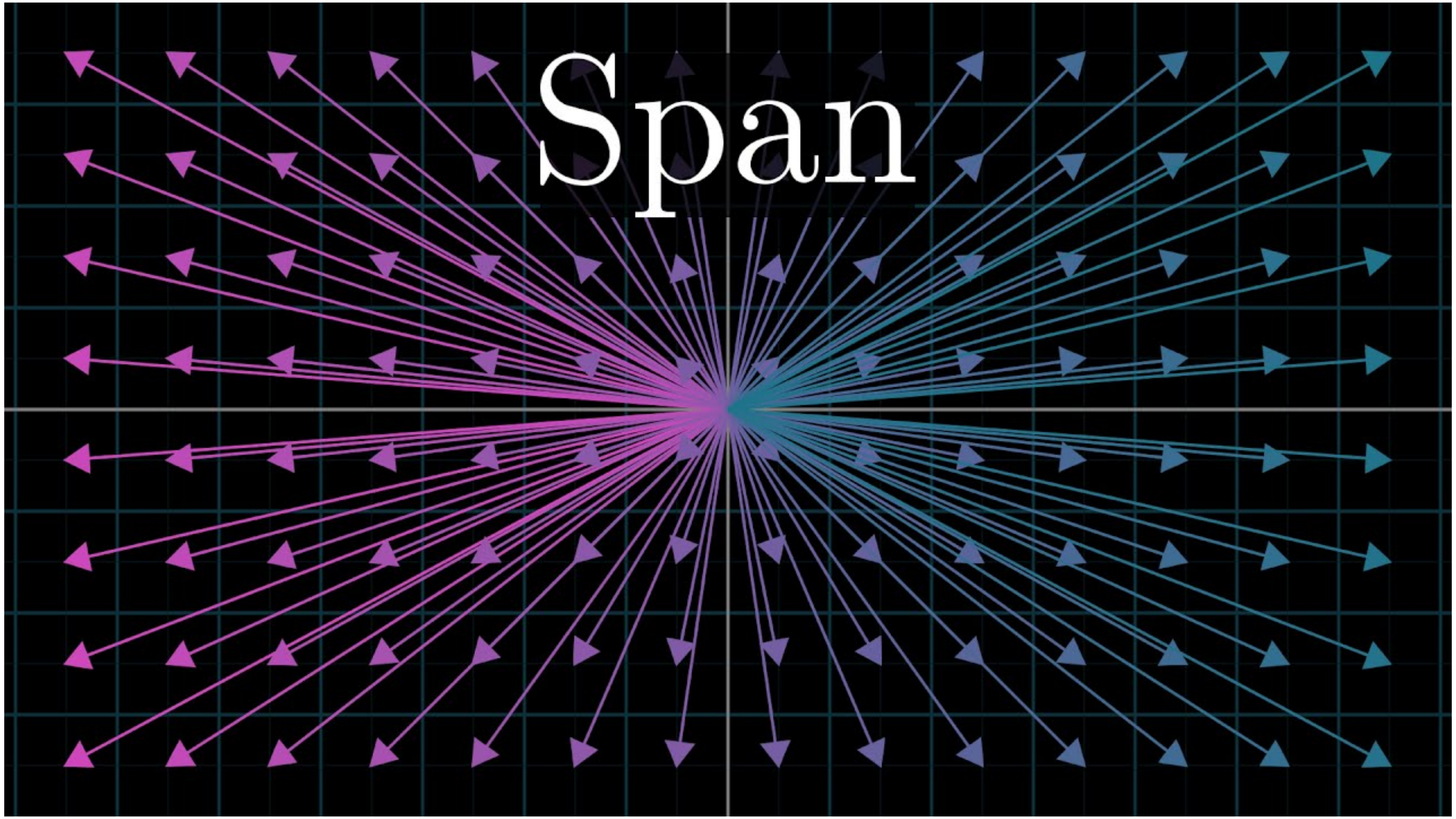

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Quick reminder: Dot Product

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(vectors need to be the same length)

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$$Ax = ?$$

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Matrix-Vector Multiplication

The Traditional Way

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \end{bmatrix}$$

Matrix-Vector Multiplication

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$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} \color{red}{1} \\ \color{teal}{1} \\ \color{green}{1} \end{bmatrix}$$

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- Alternative way to think about this multiplication
  - The matrix consists of **column vectors**

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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  - The vector provides the **constants** for a **linear combination of the columns**

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    - The matrix consists of **column vectors**
    - The vector provides the **constants** for a **linear combination of the columns**

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} \color{red}{1} \\ \color{teal}{1} \\ \color{green}{1} \end{bmatrix} \longrightarrow \color{red}{1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \color{teal}{1} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \color{green}{1} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 18 \end{bmatrix}$$

Matrix-Vector Multiplication

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- What can you tell about the Column Space of this matrix?

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  - **However**, the function's range **may not span** R^M, unless it is **rank M**

Linear Transformations

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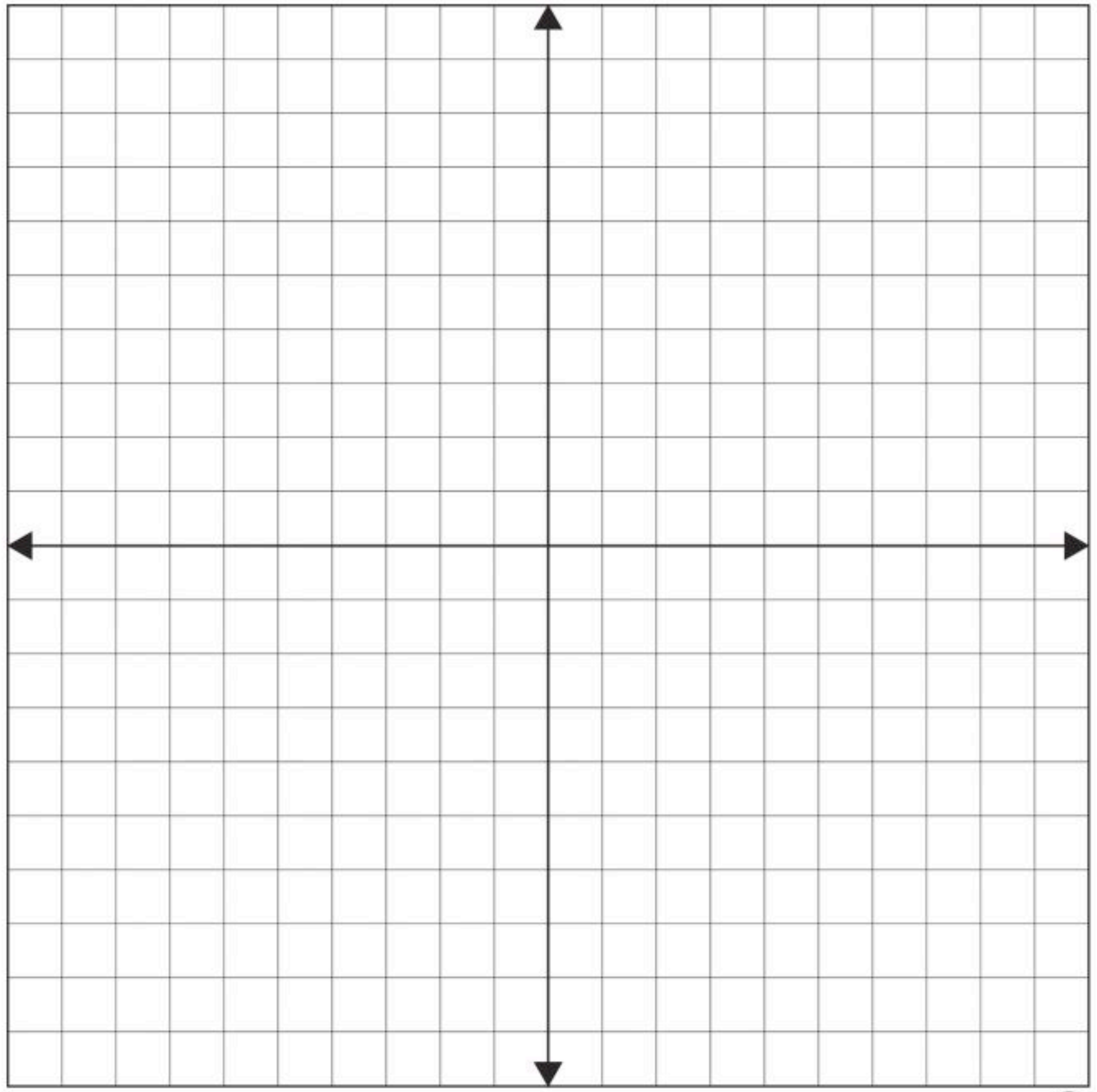
Identity Matrix as a Basis

$$\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\end{bmatrix}$$



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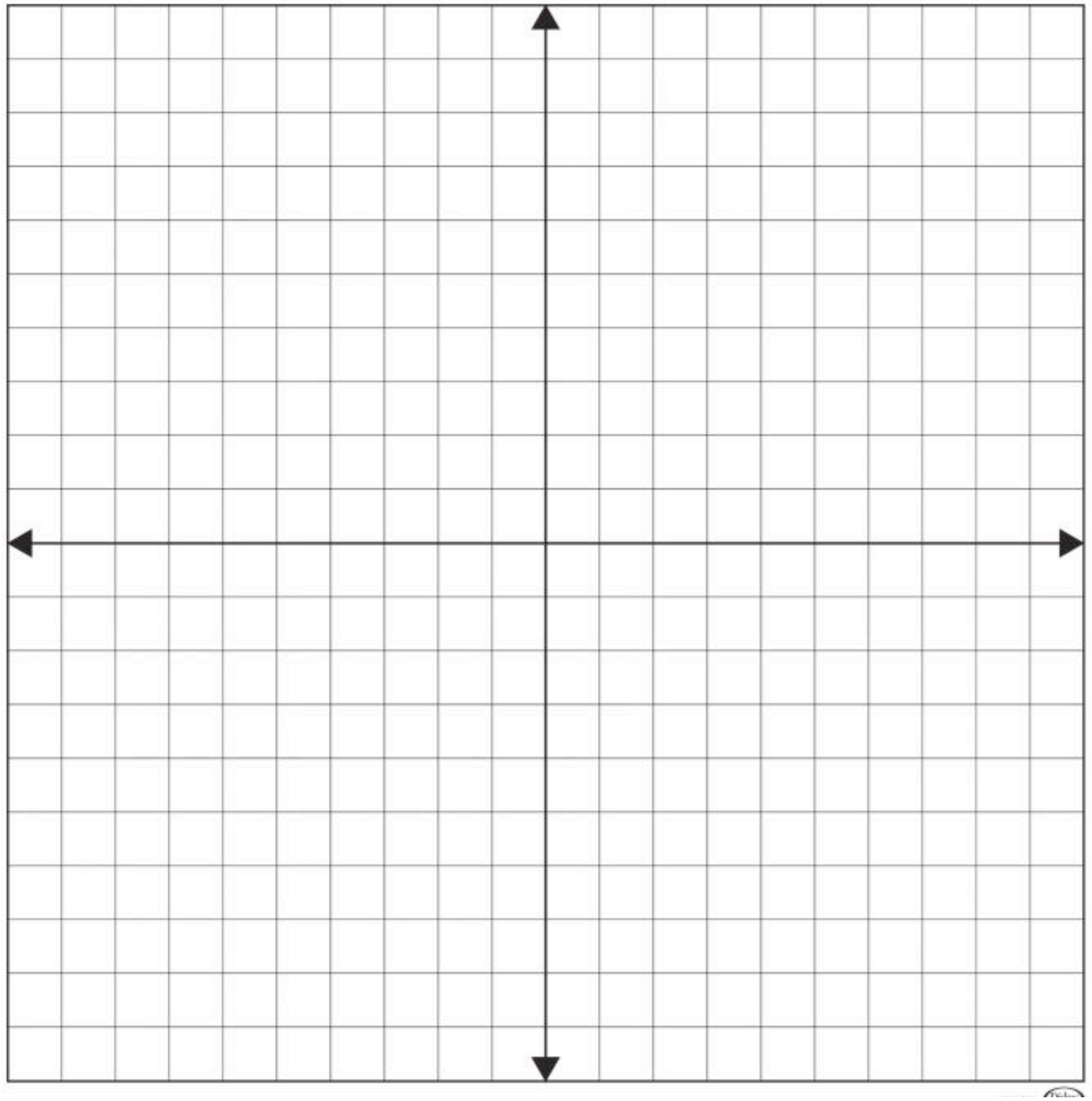

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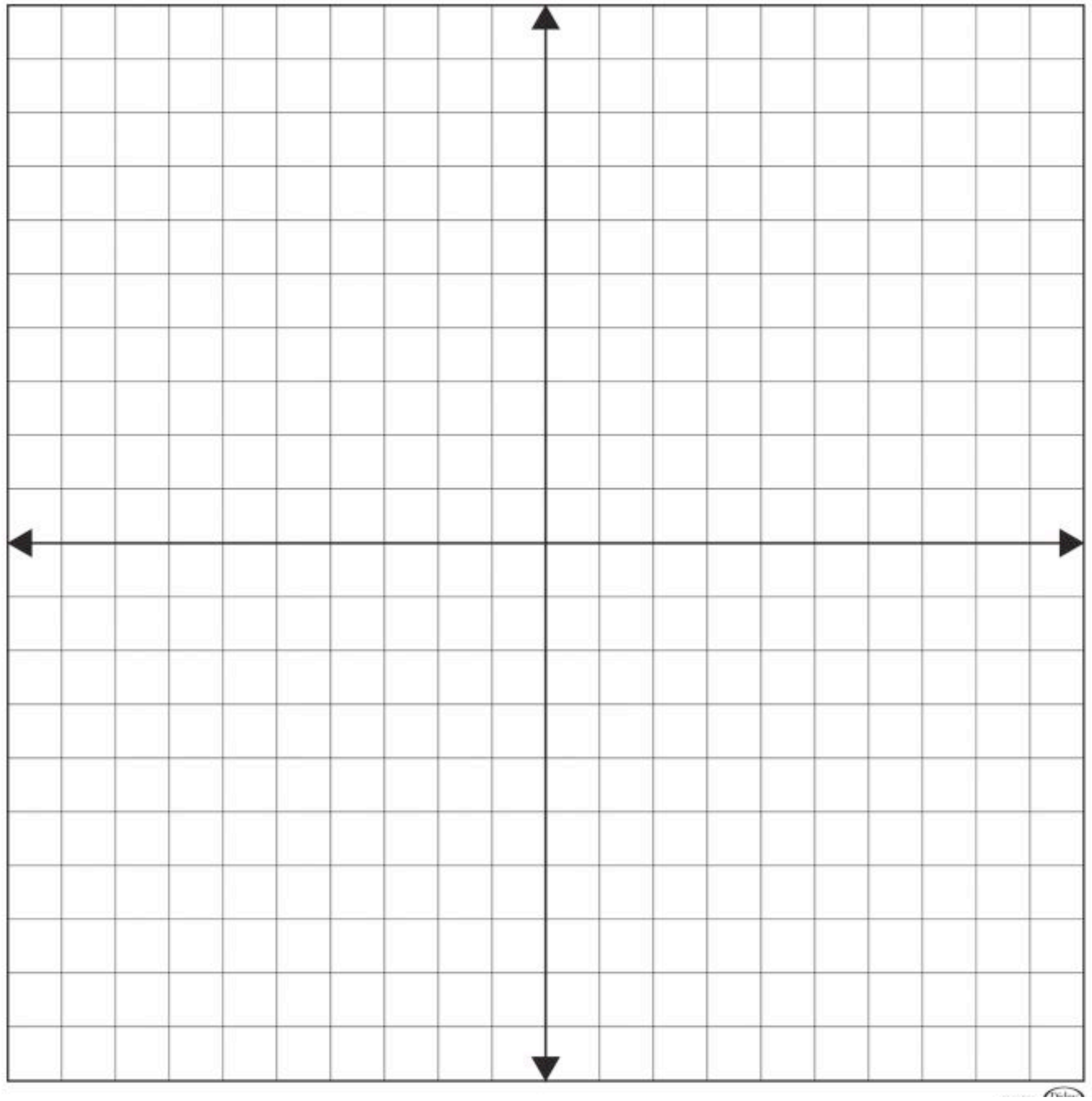

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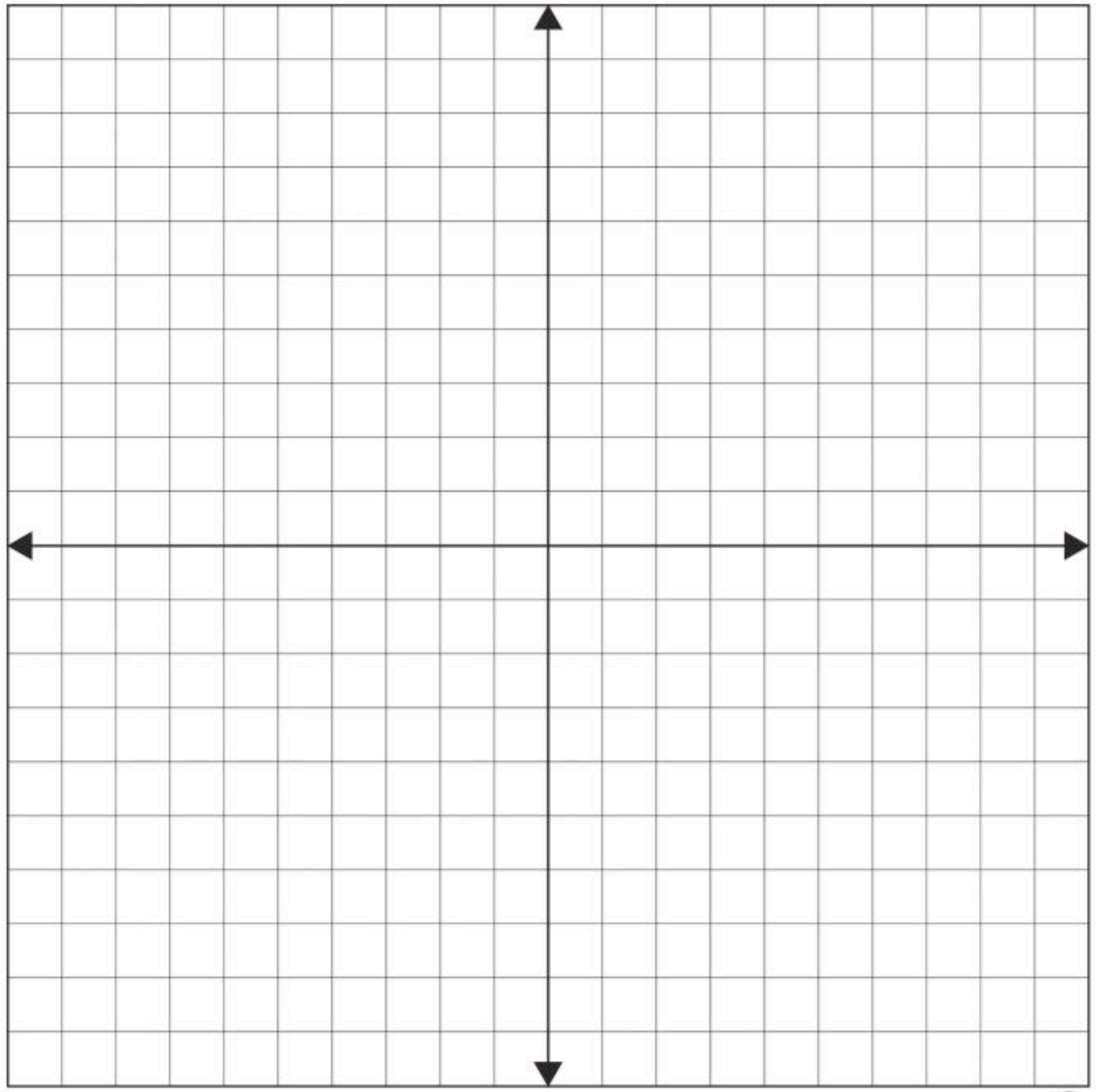

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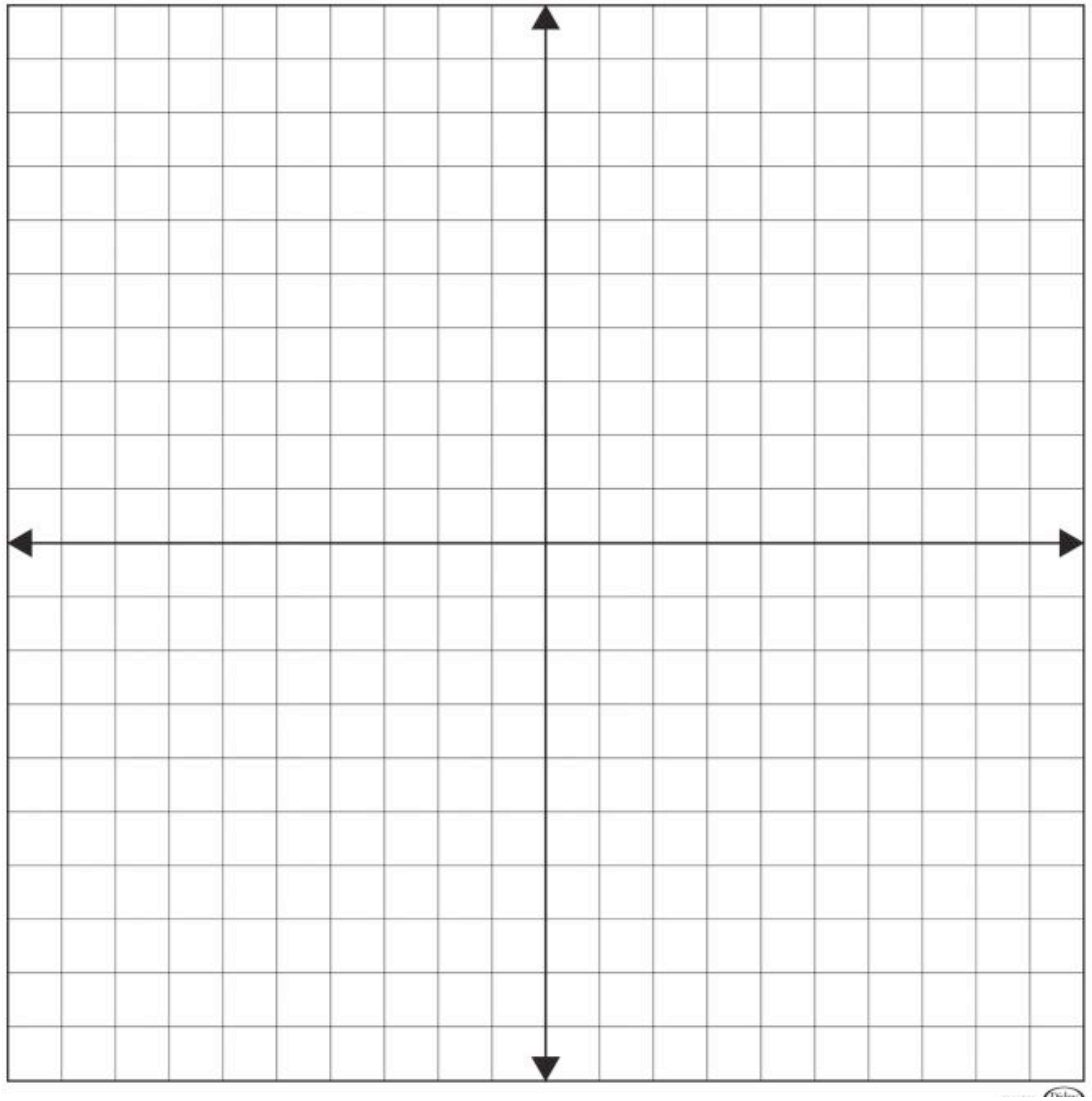

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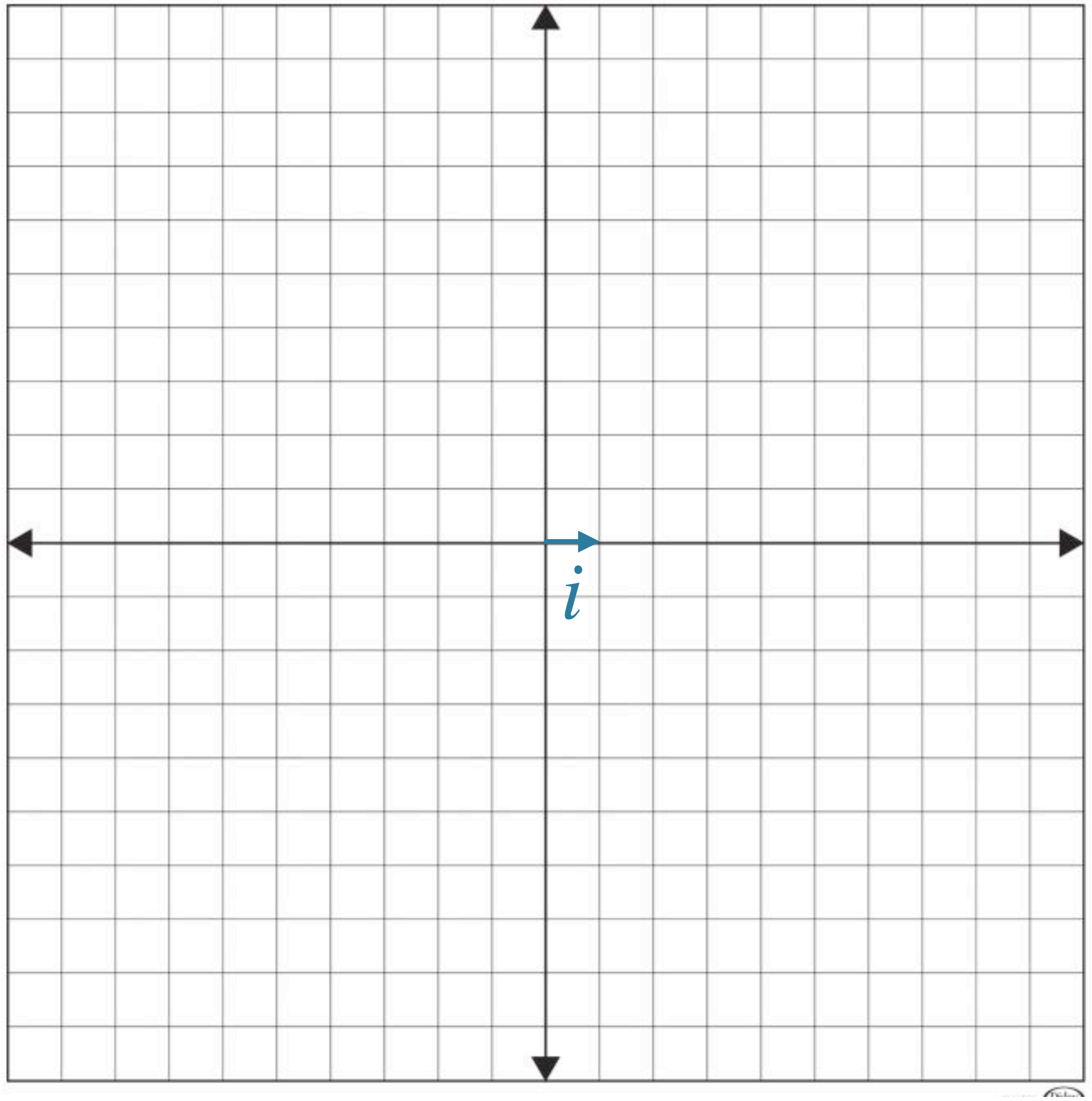

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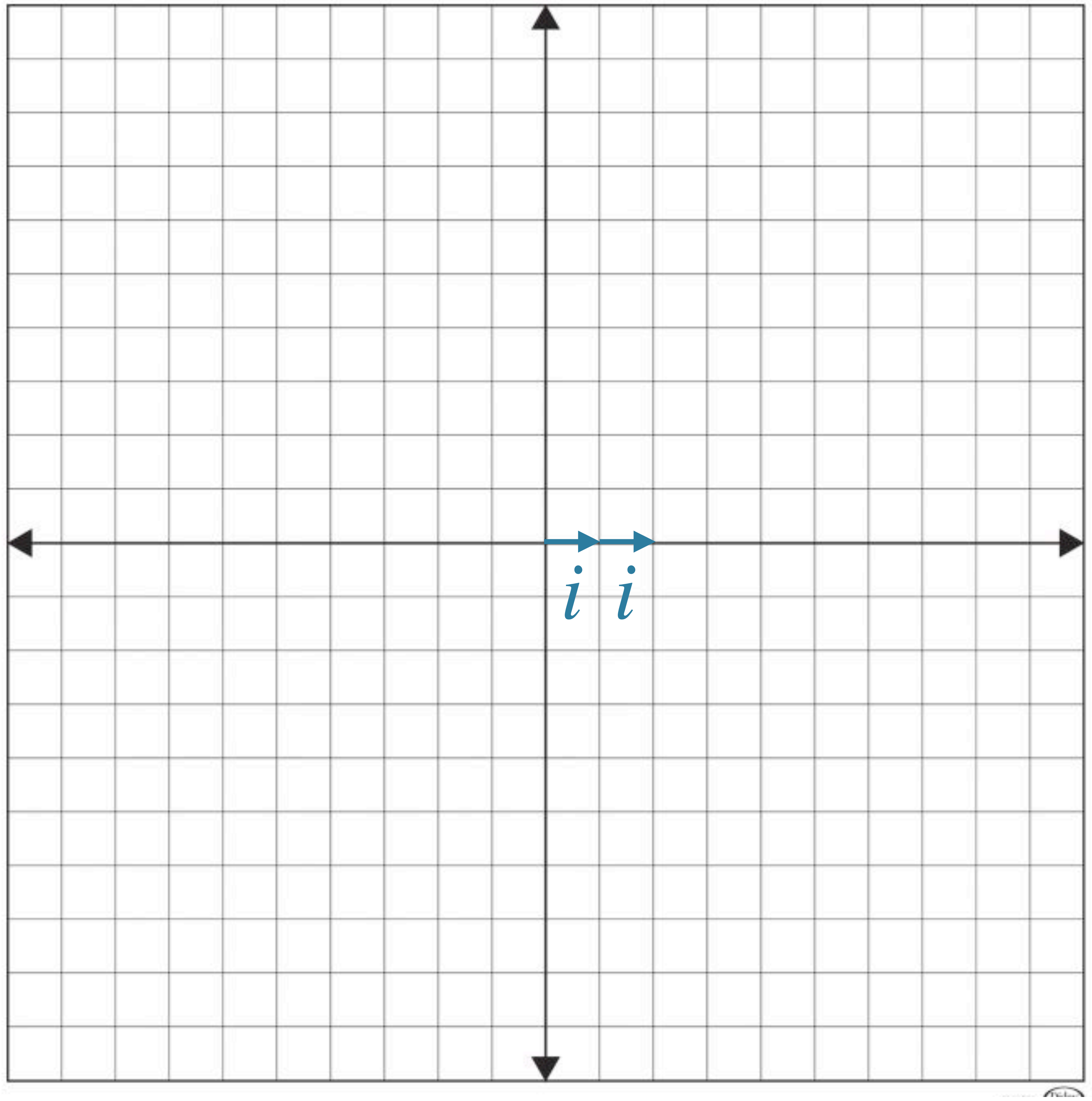

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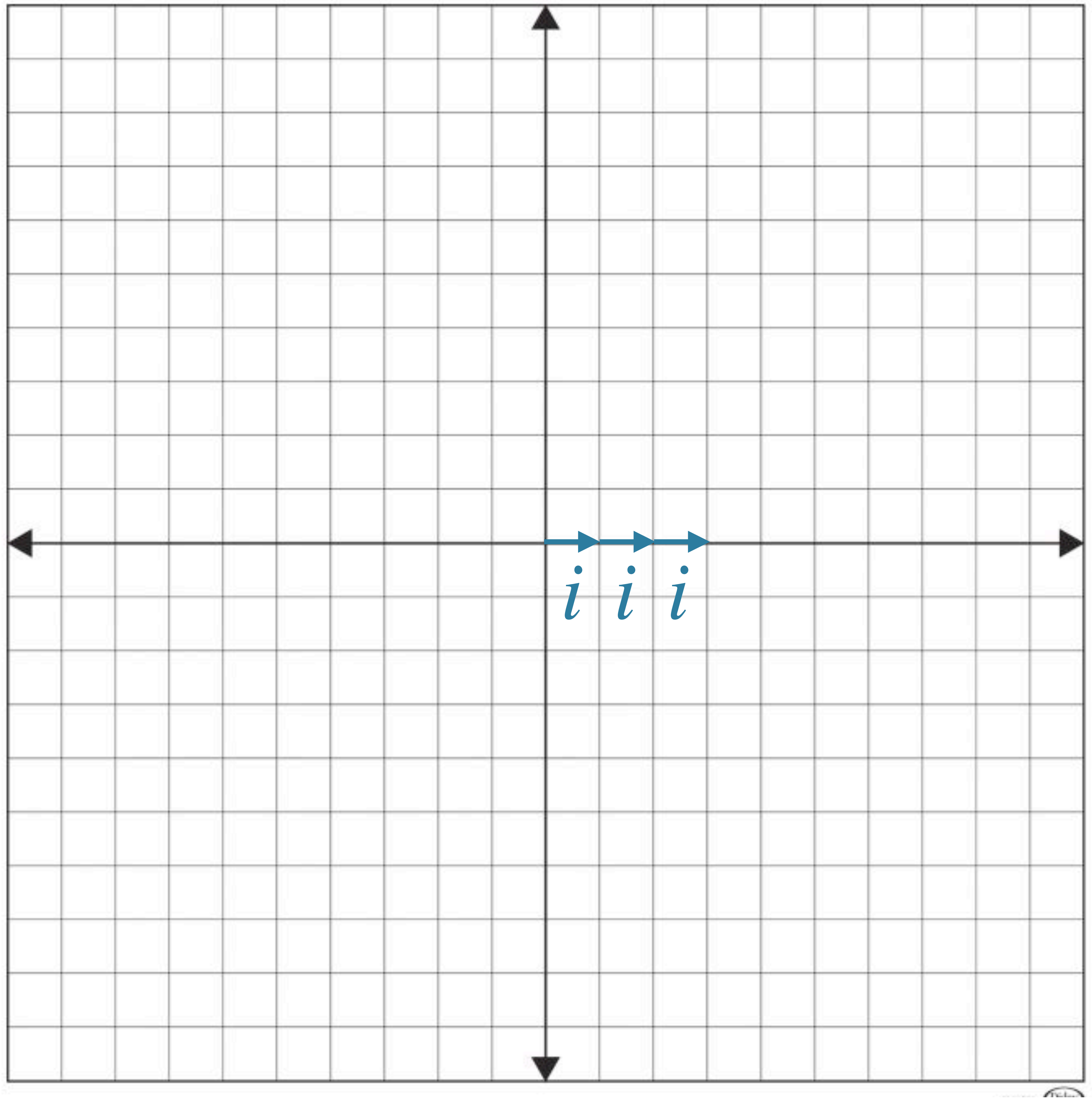

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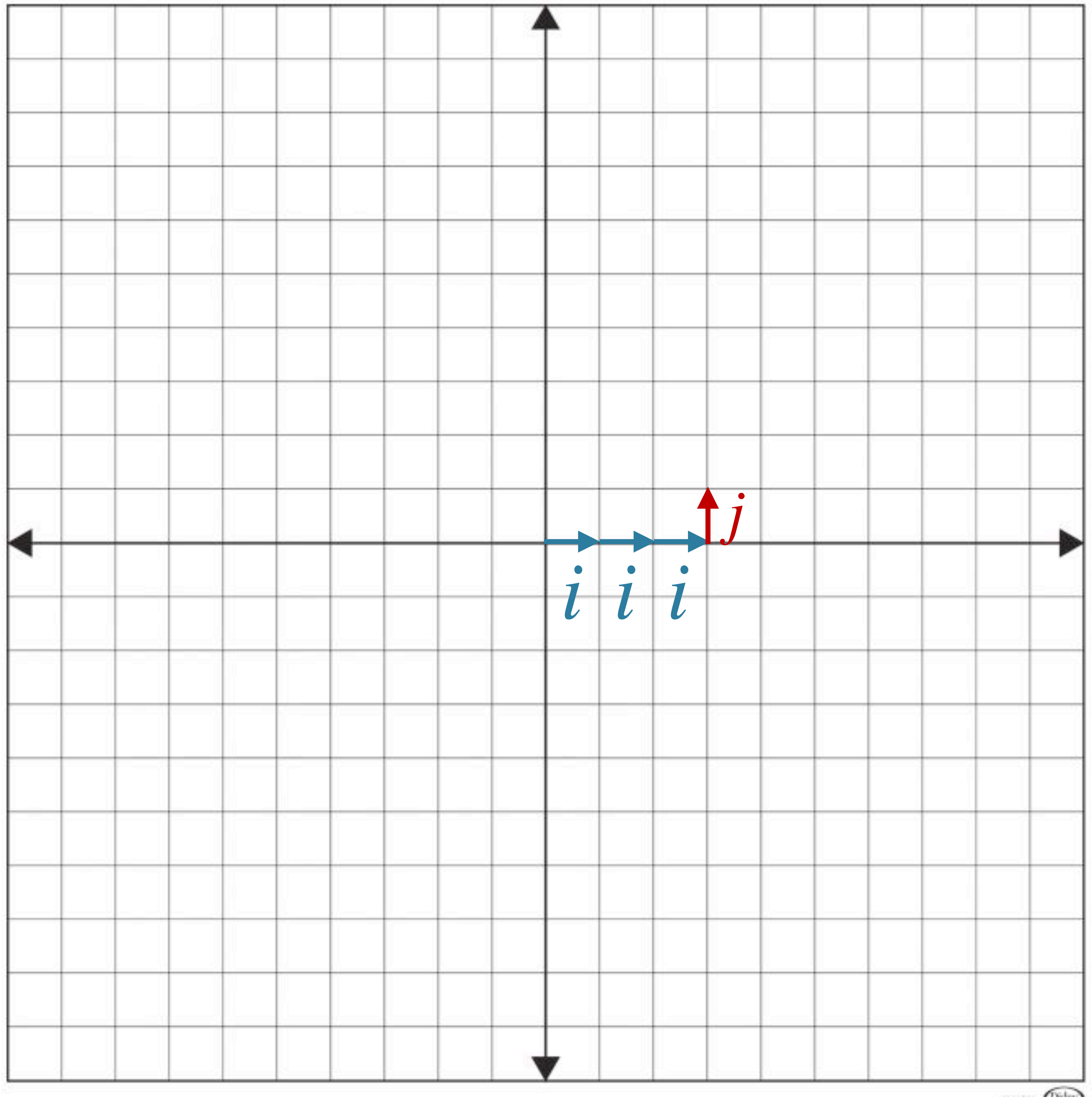

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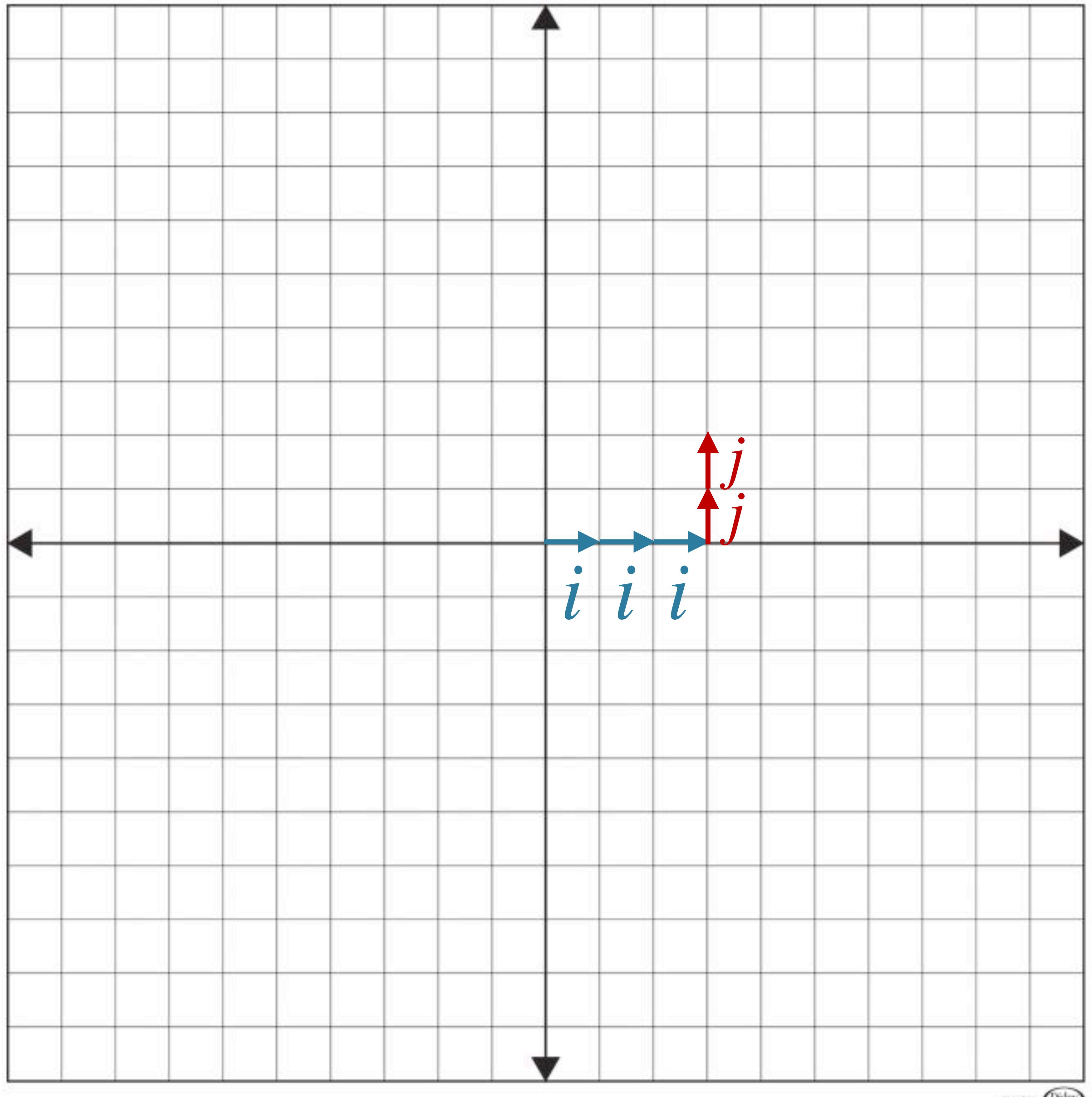

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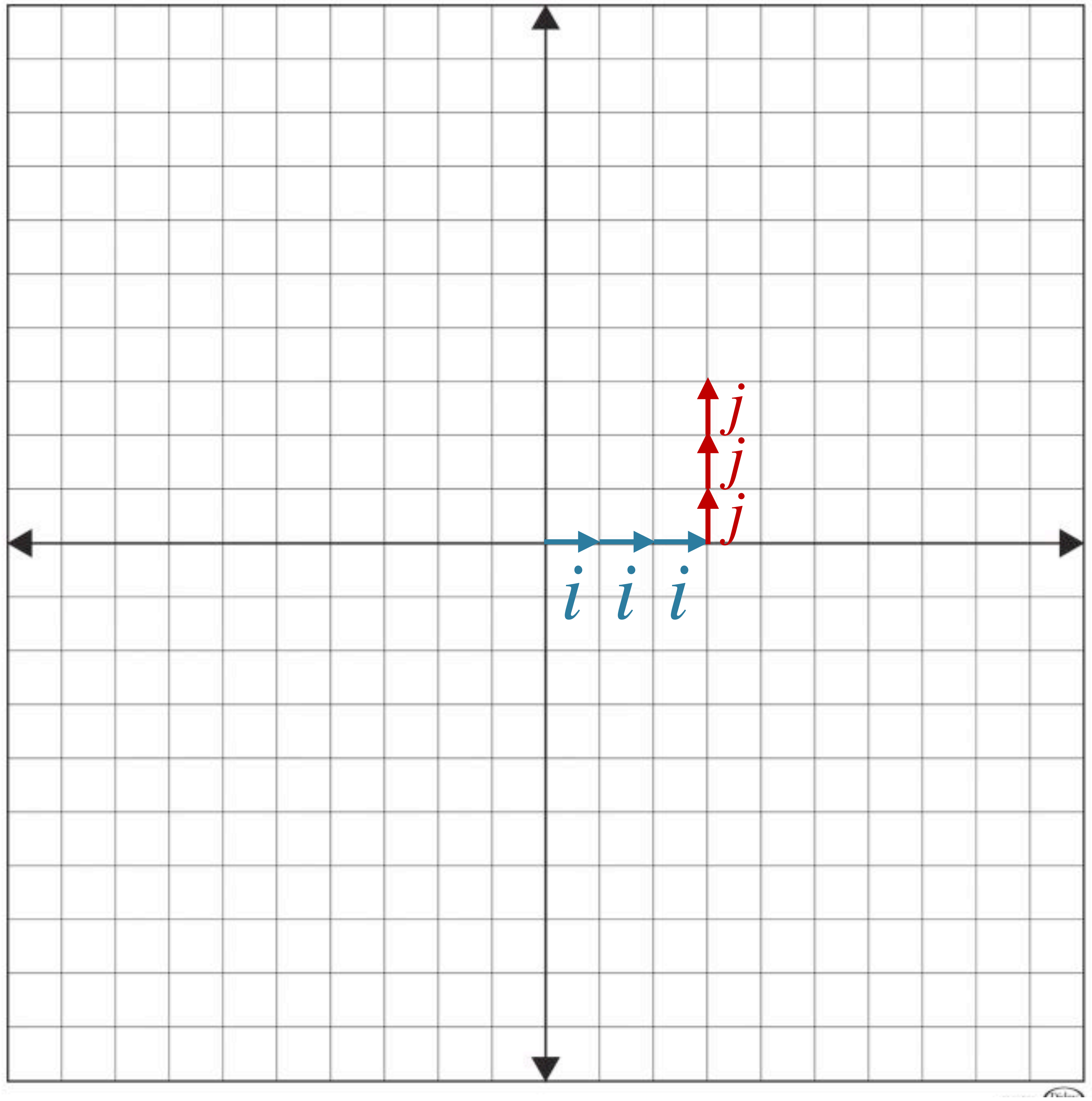

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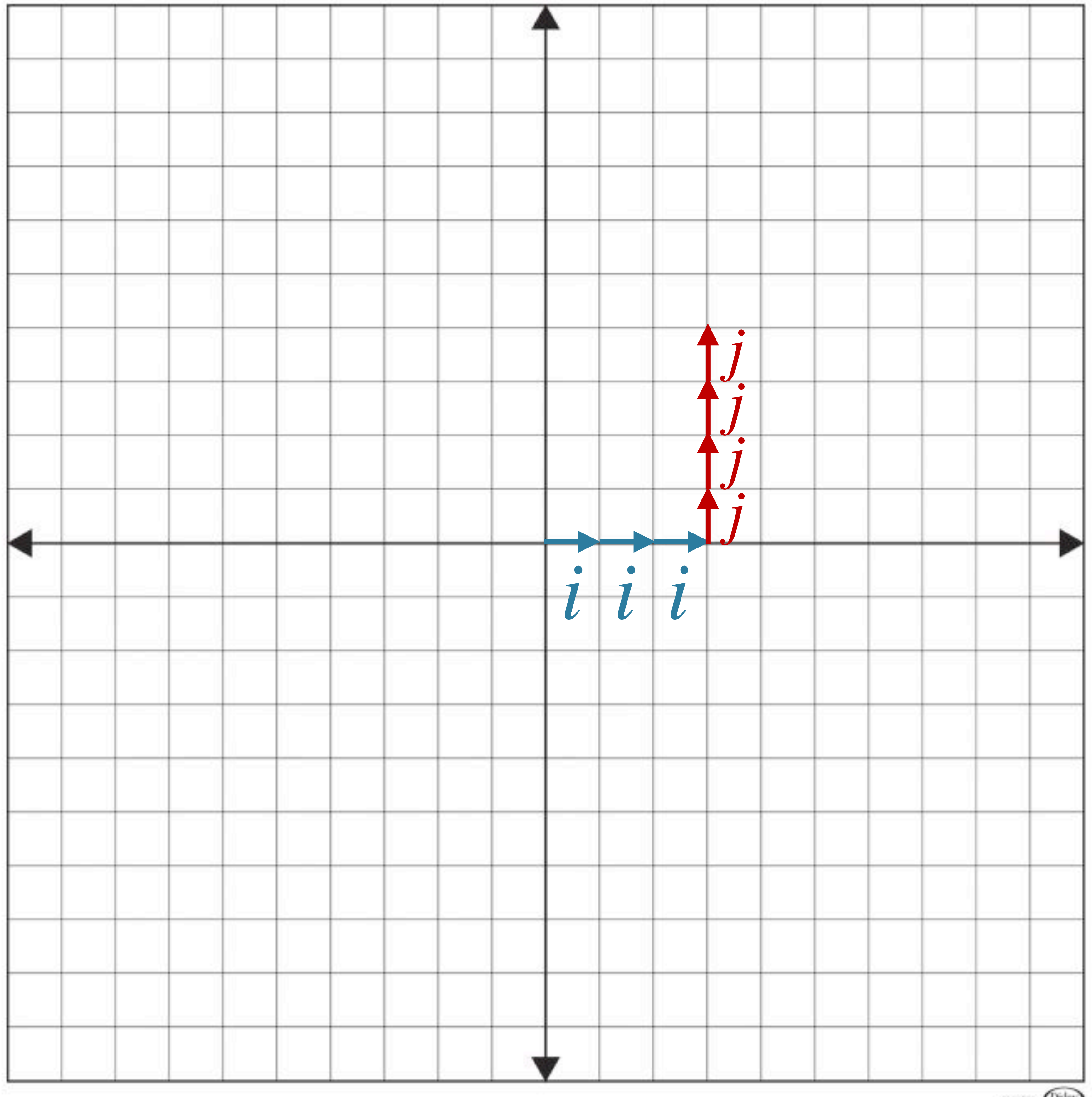

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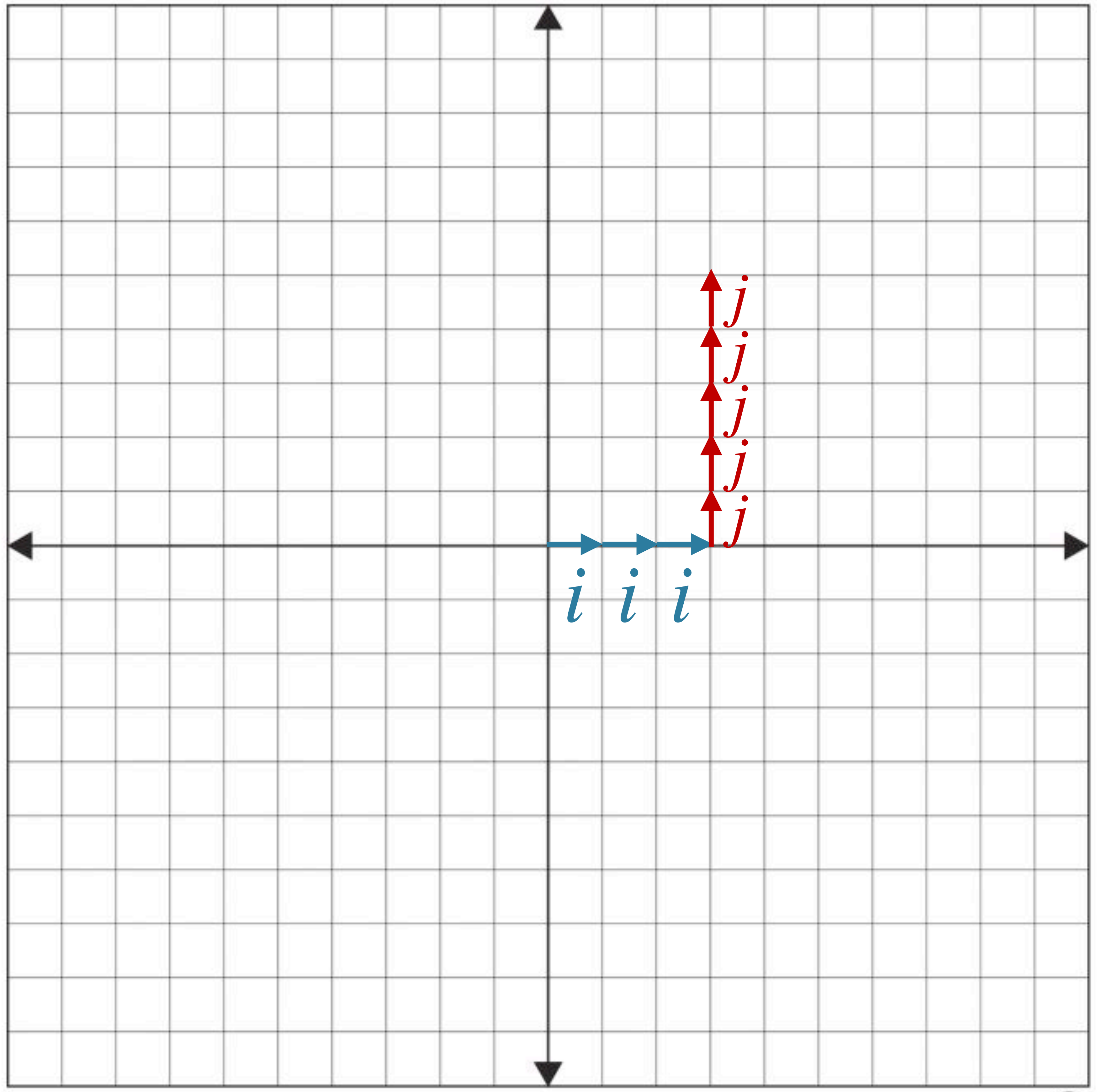

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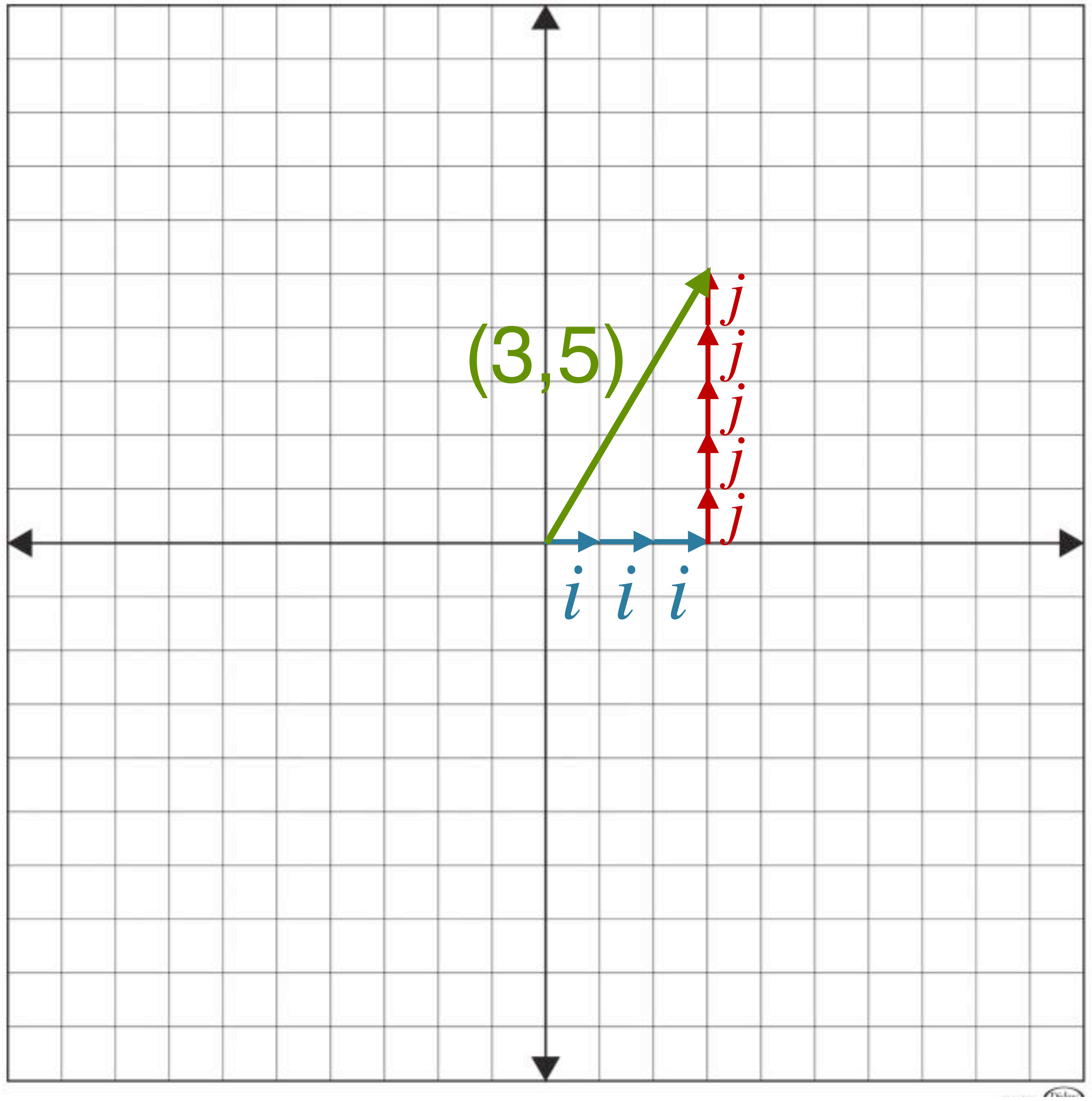

Linear Transformation

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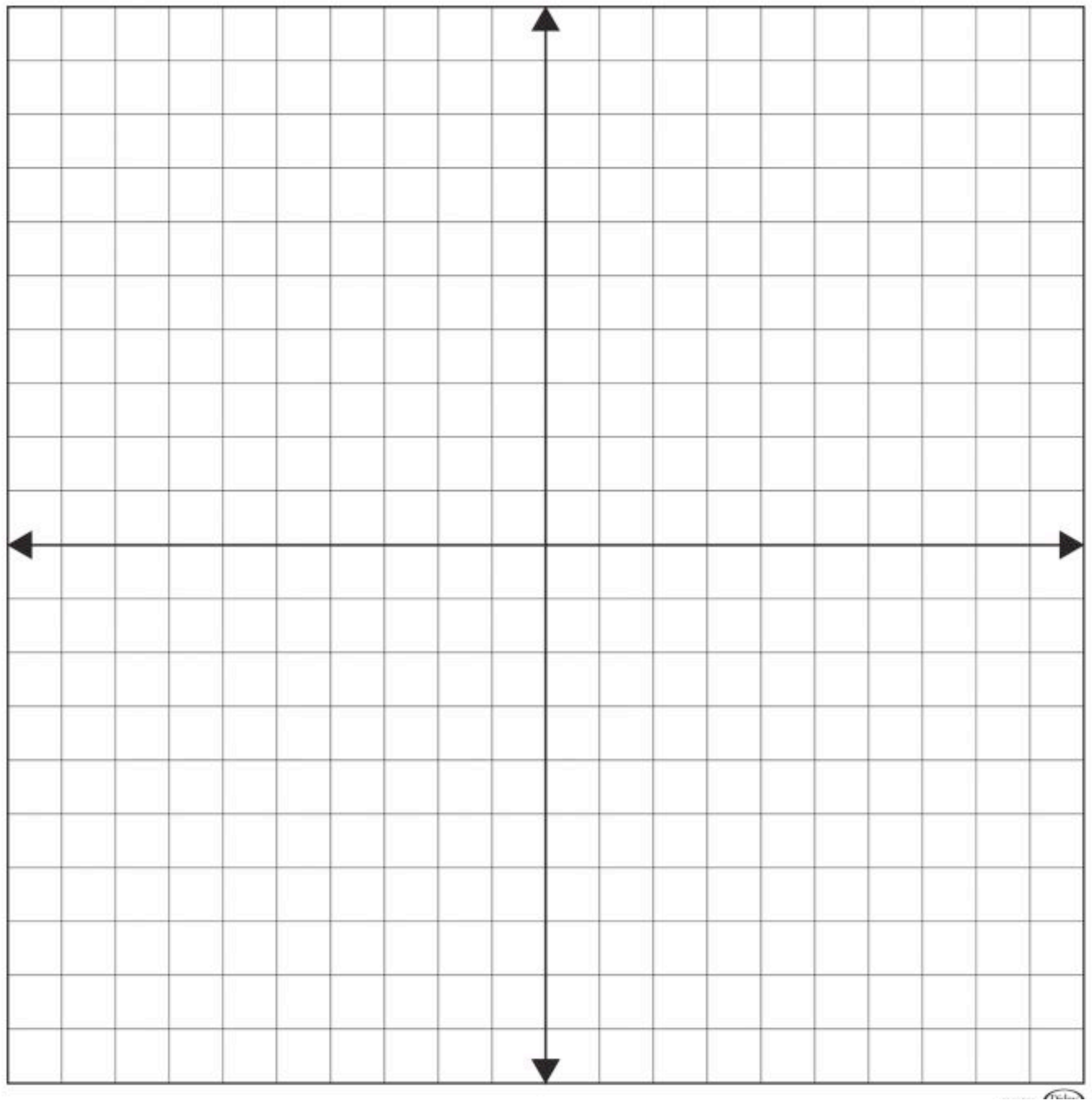

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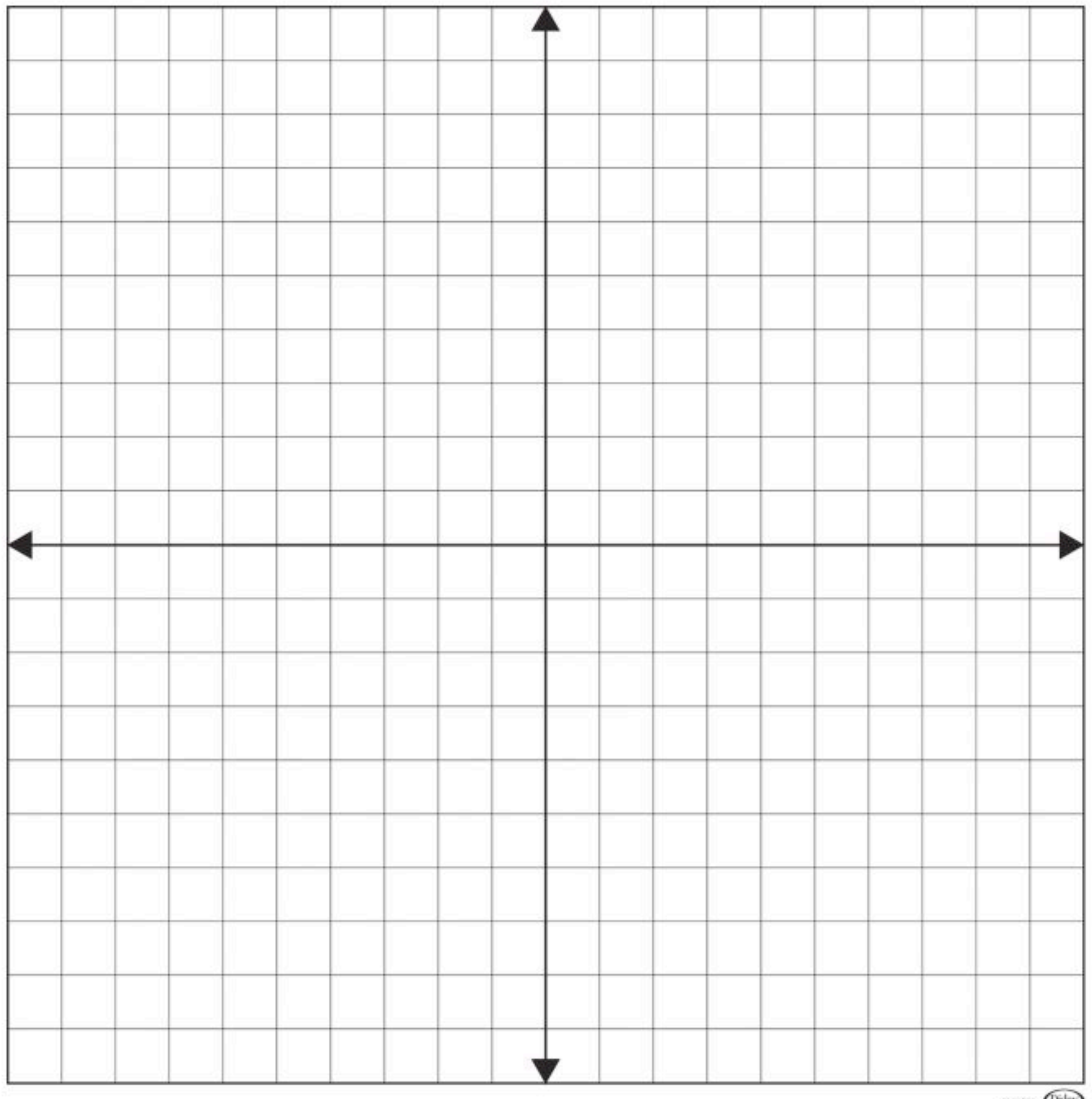

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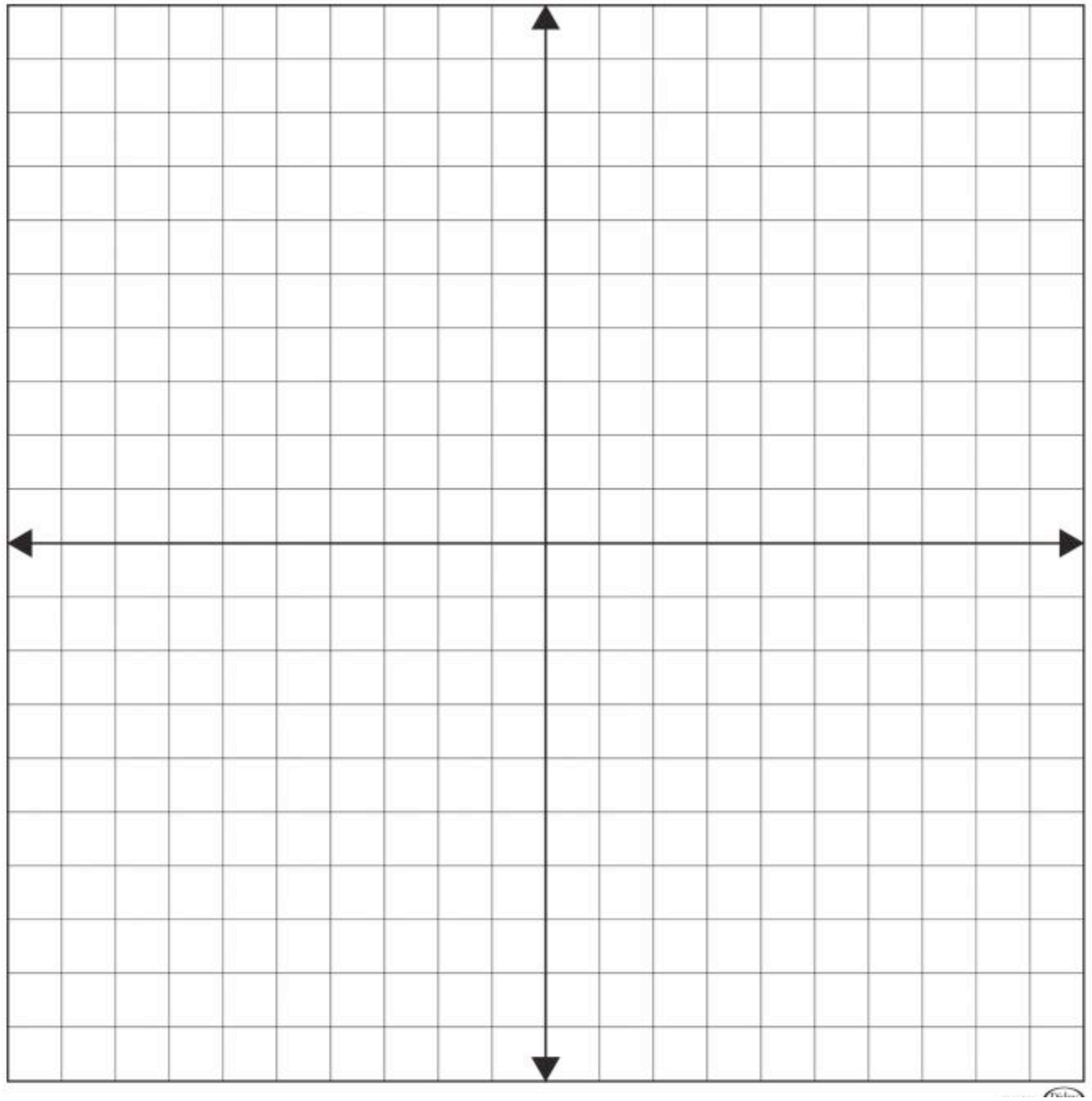

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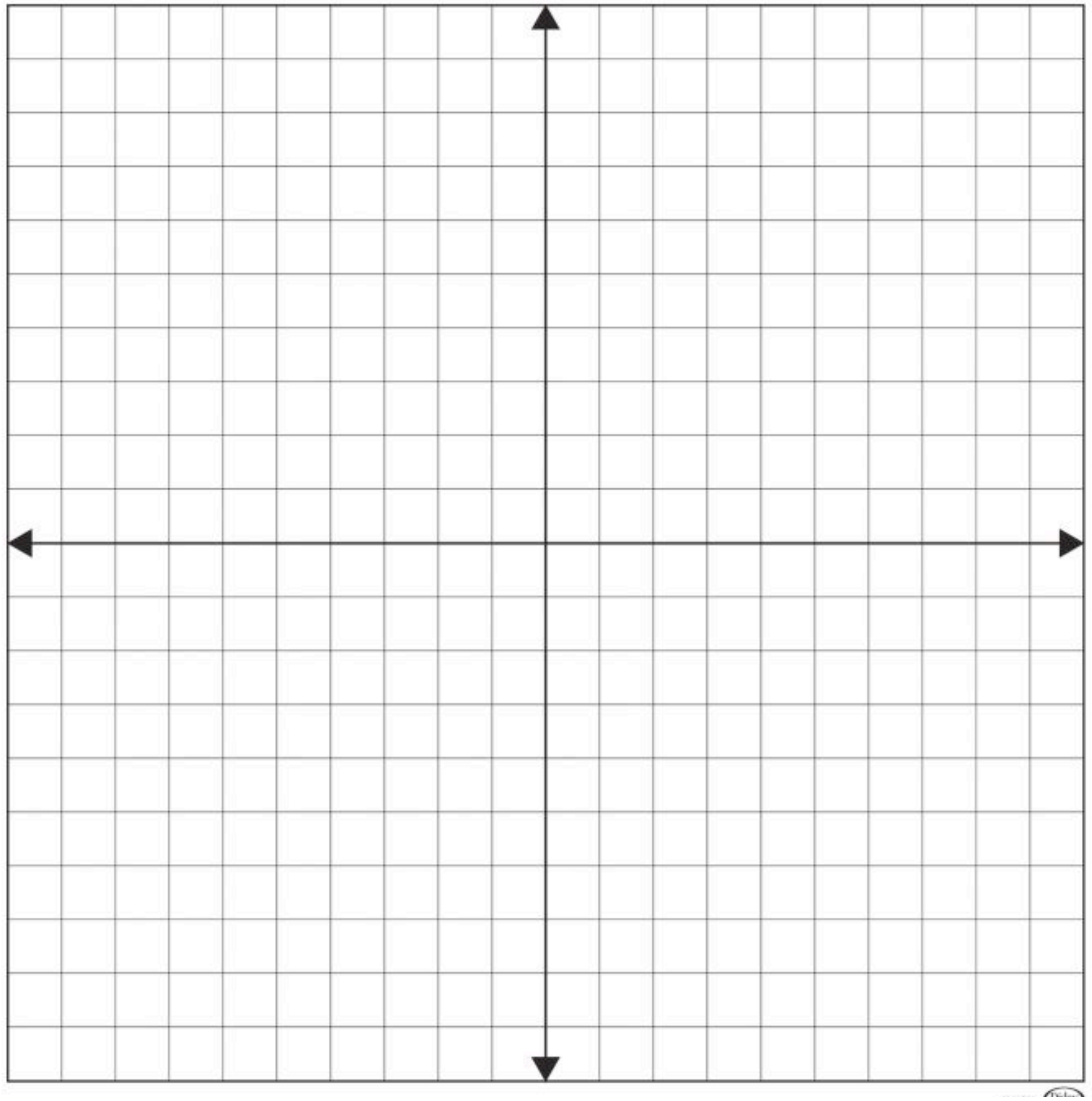

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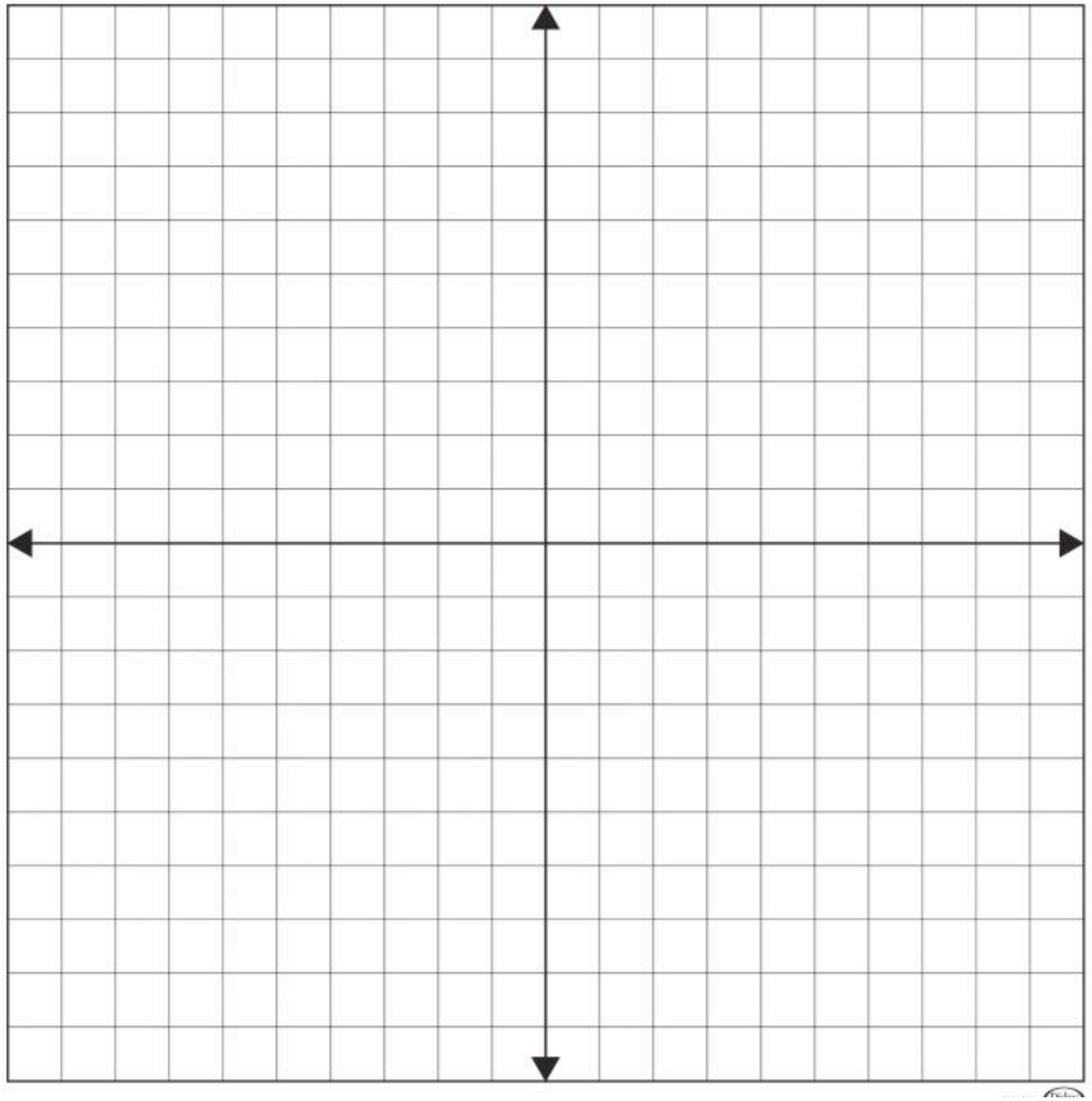

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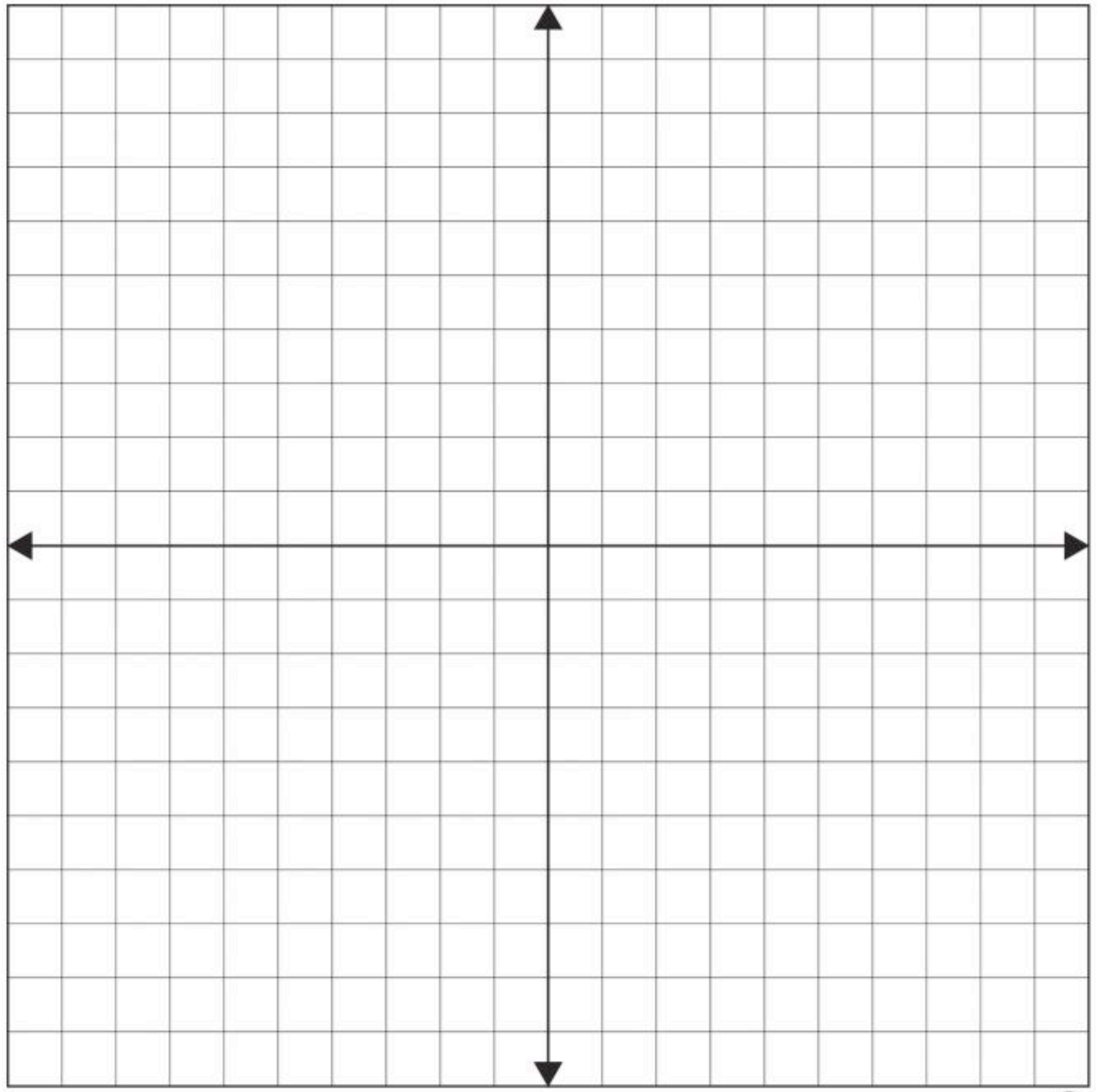

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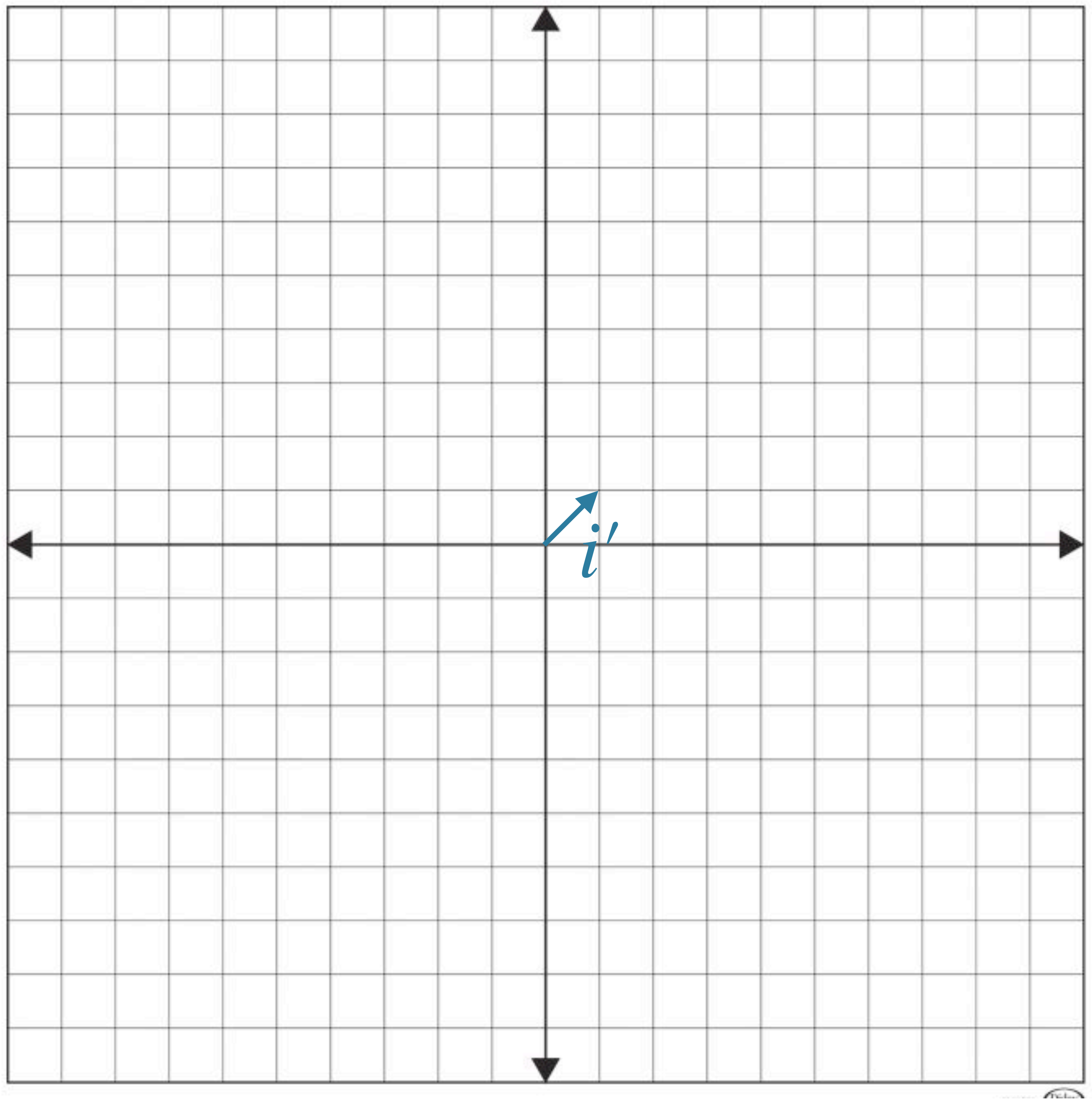

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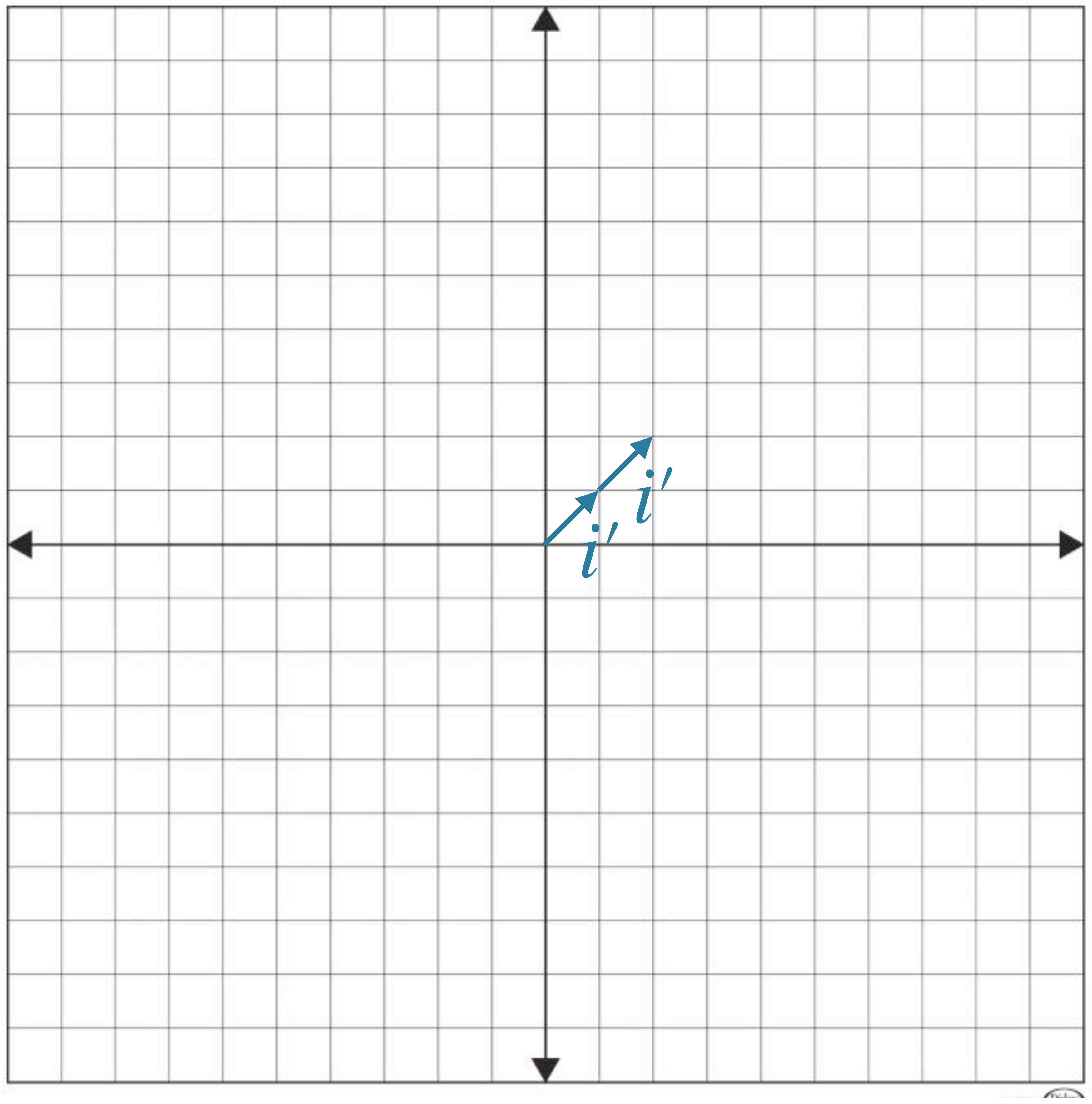

Linear Transformation

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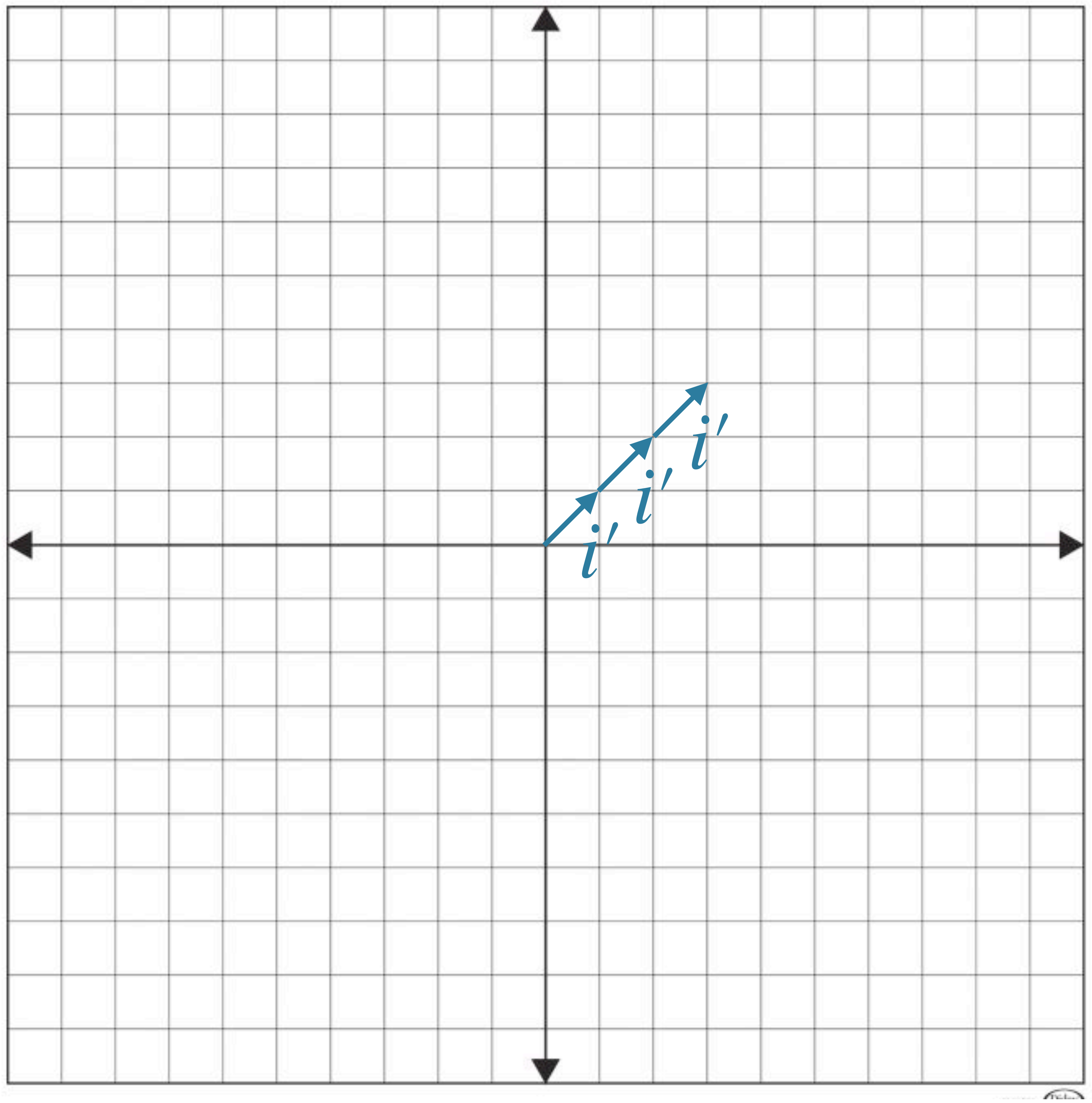

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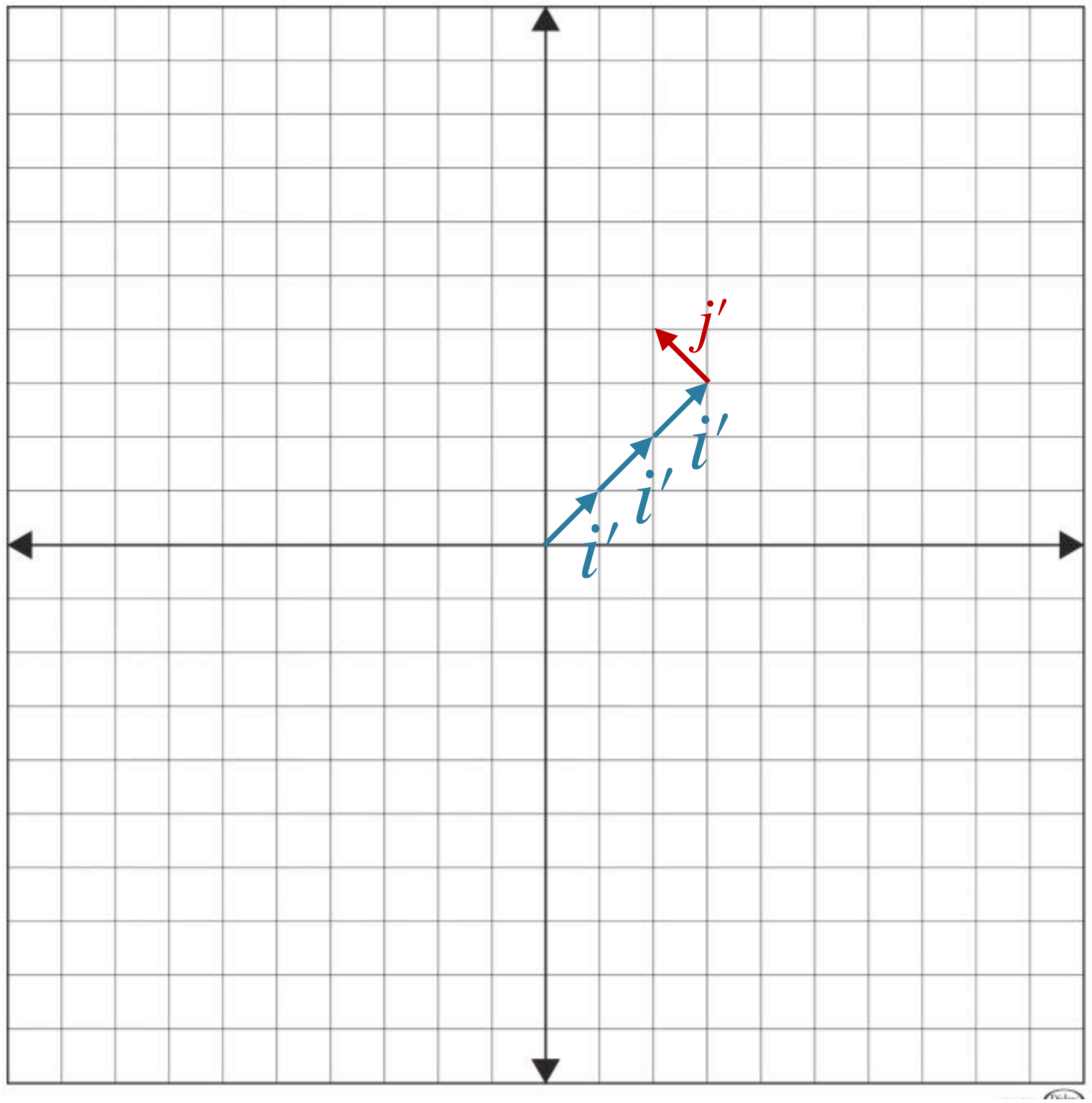

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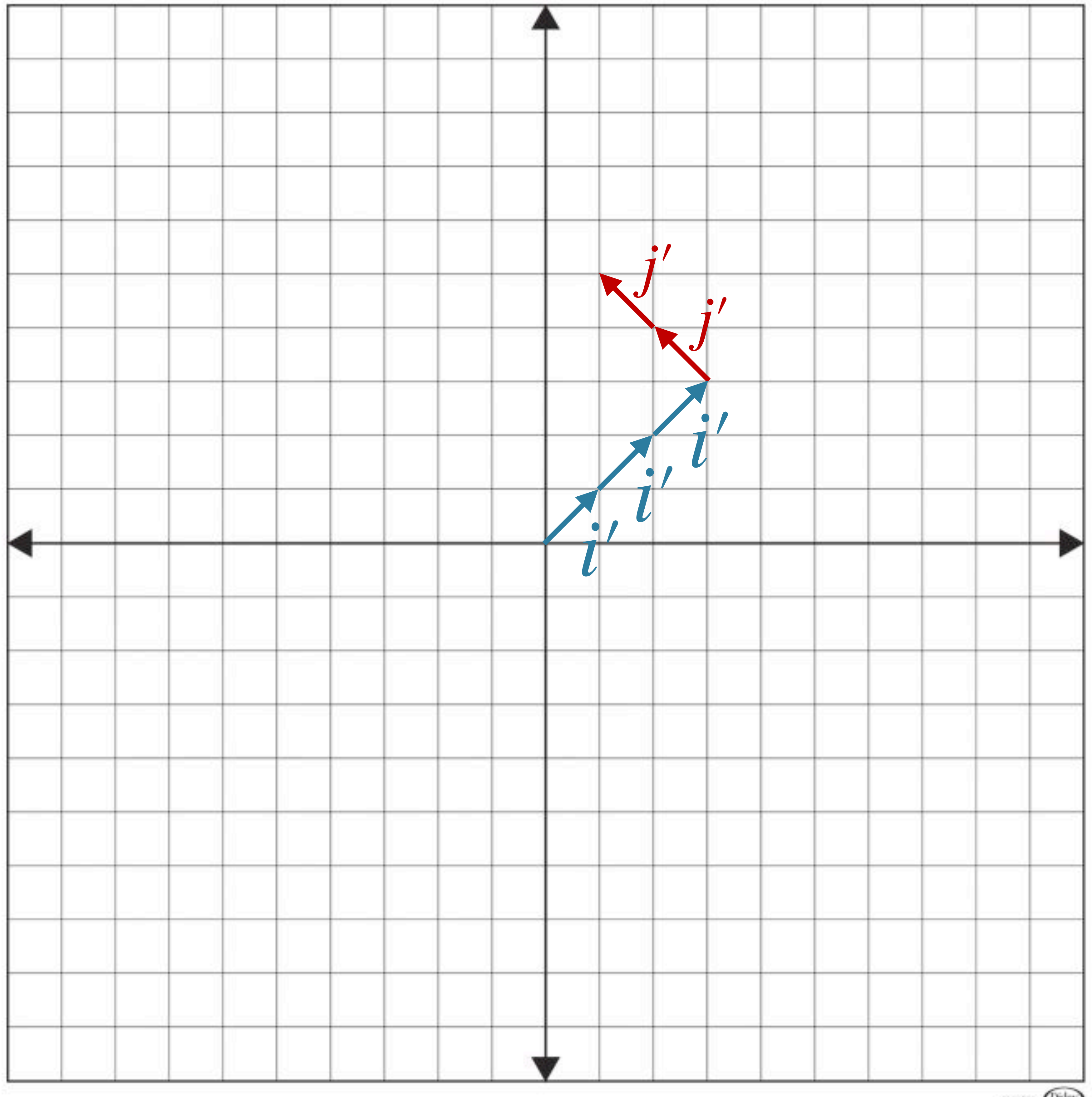

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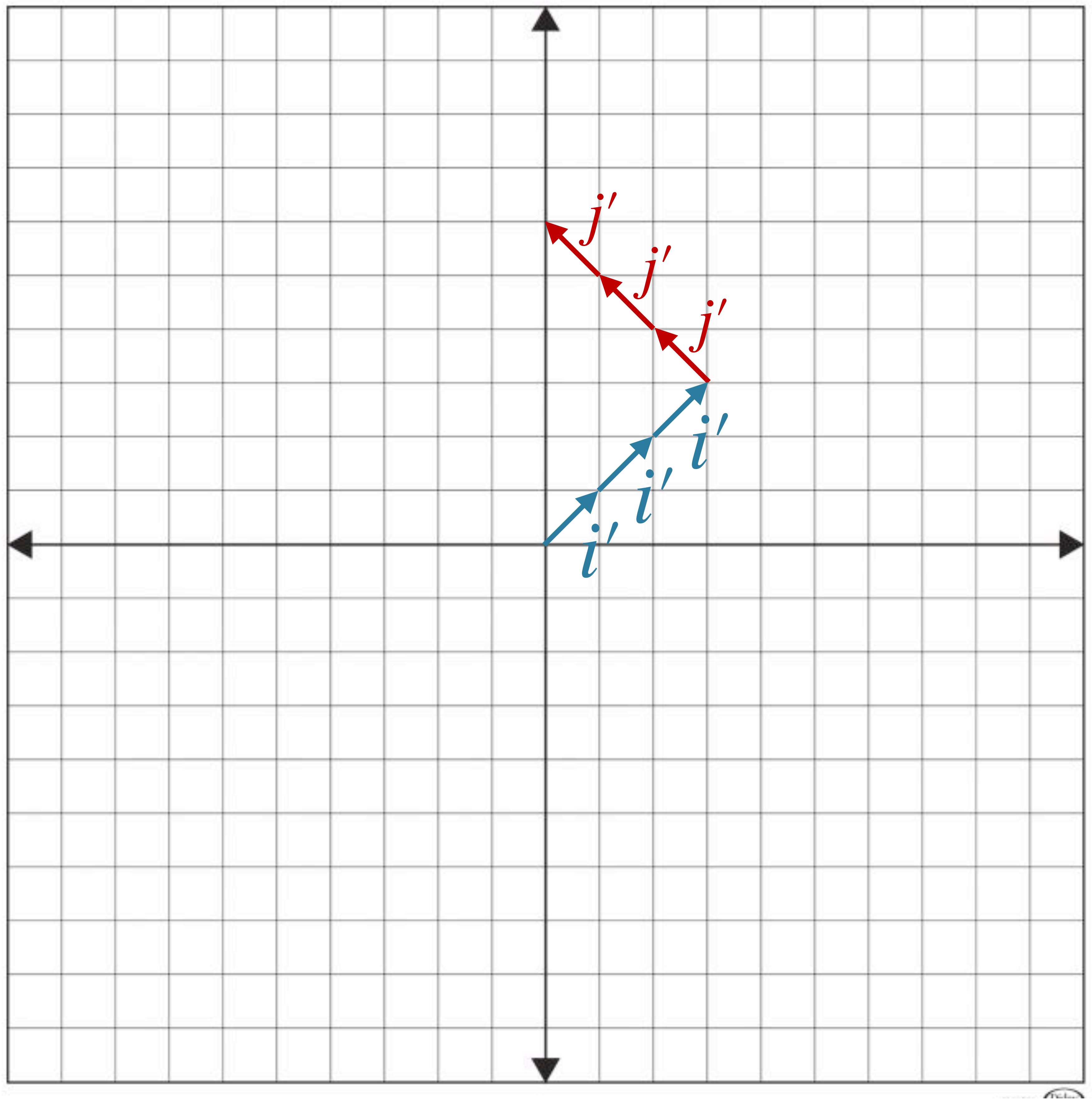

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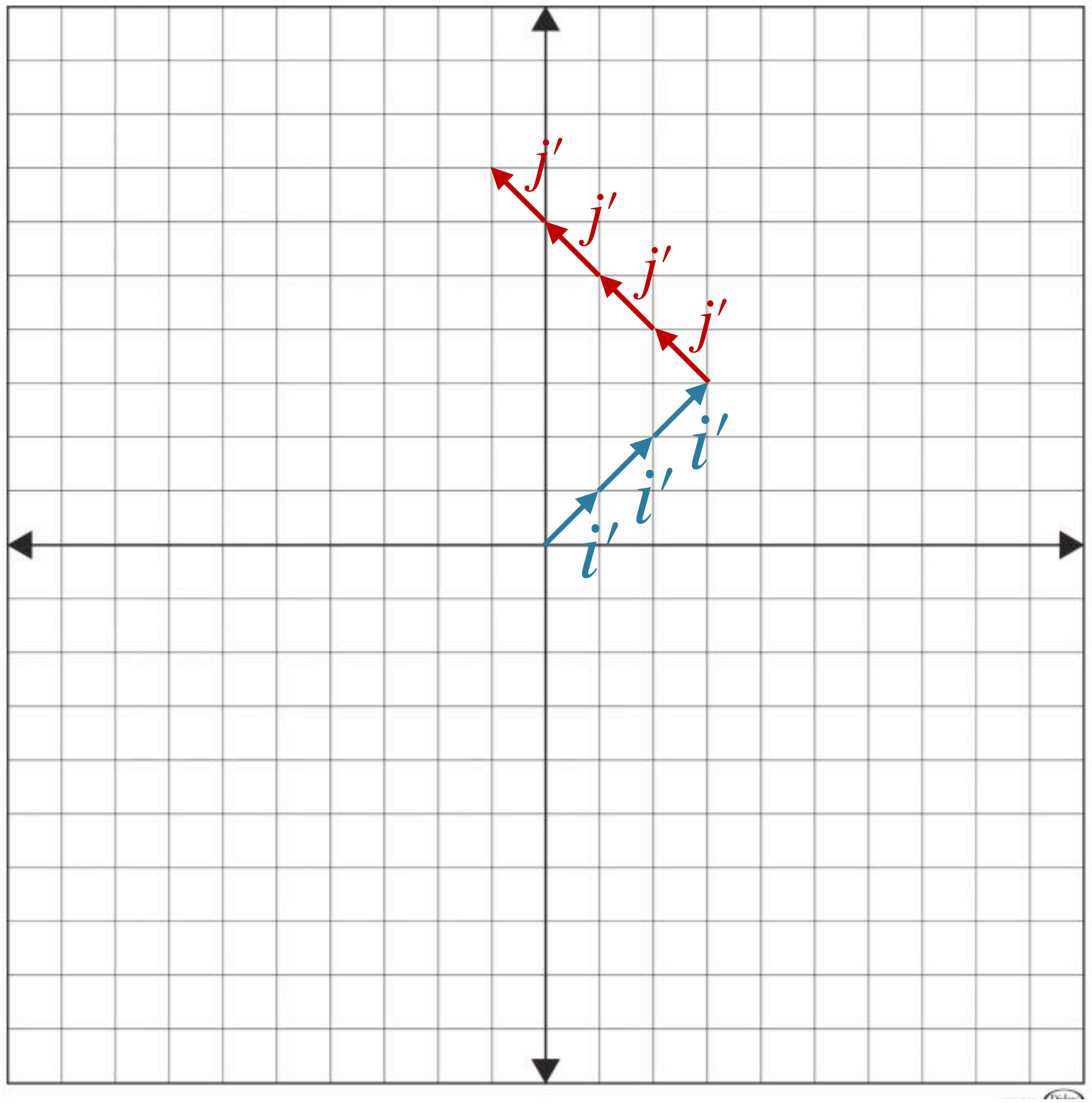

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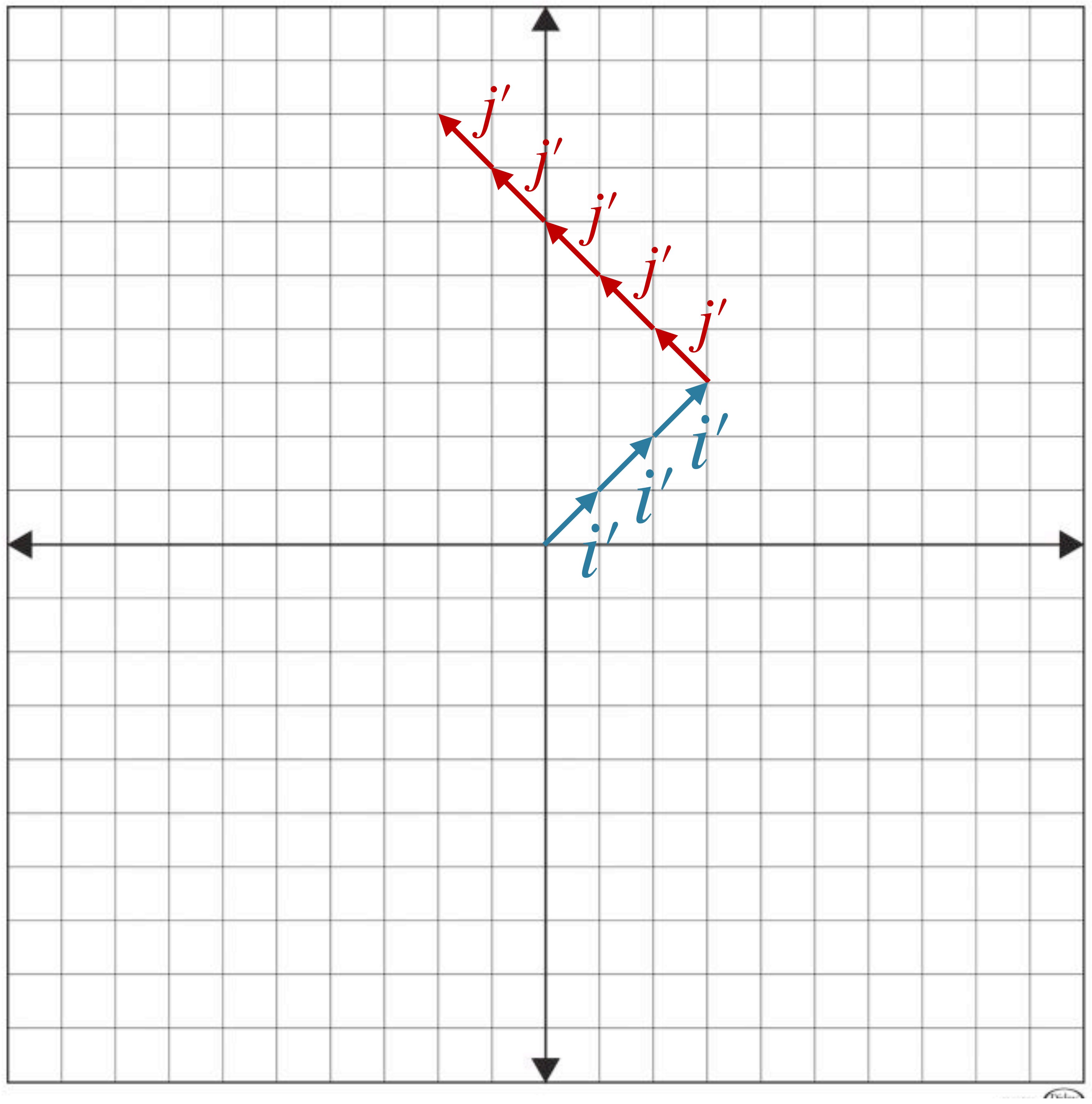

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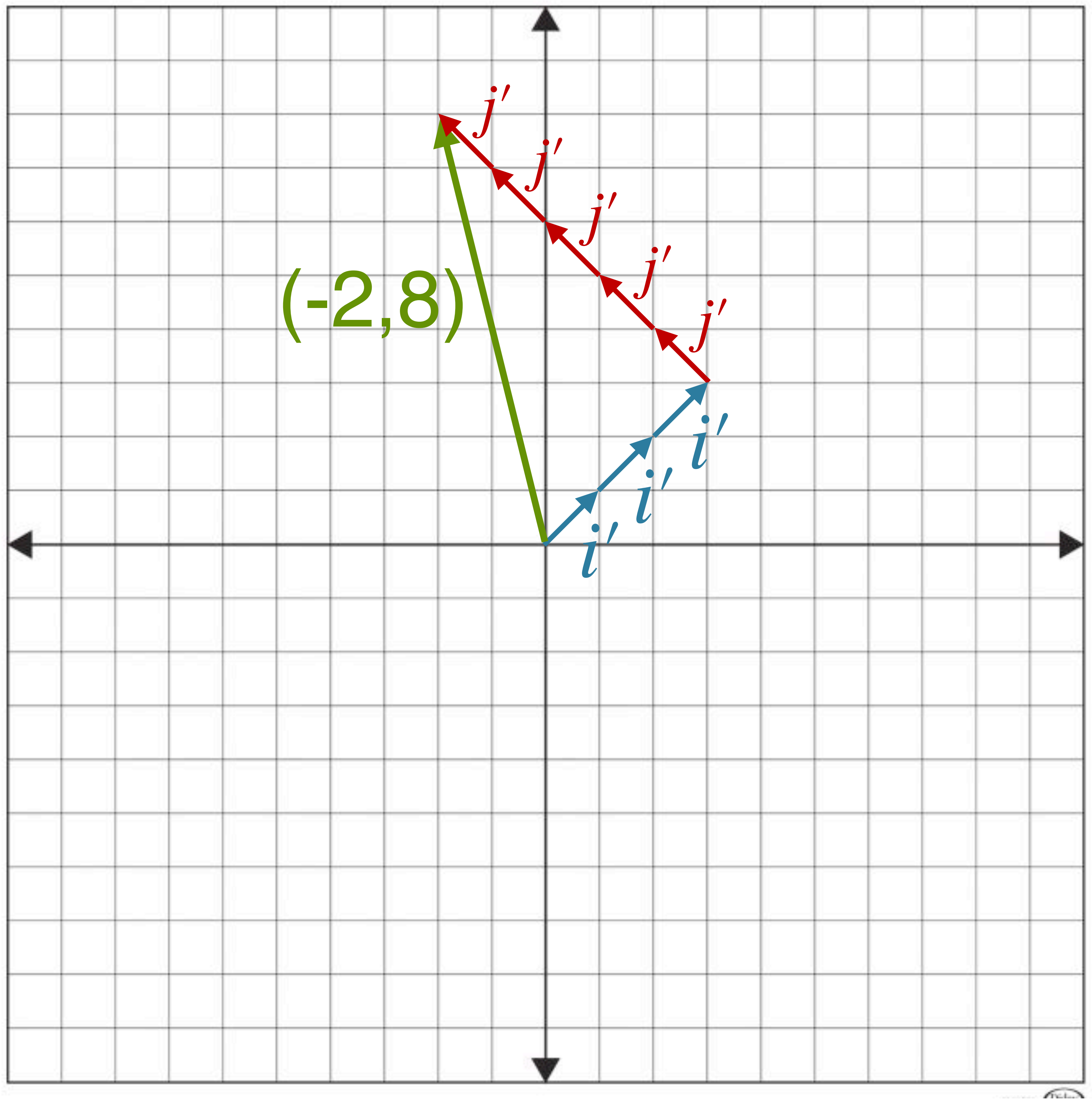

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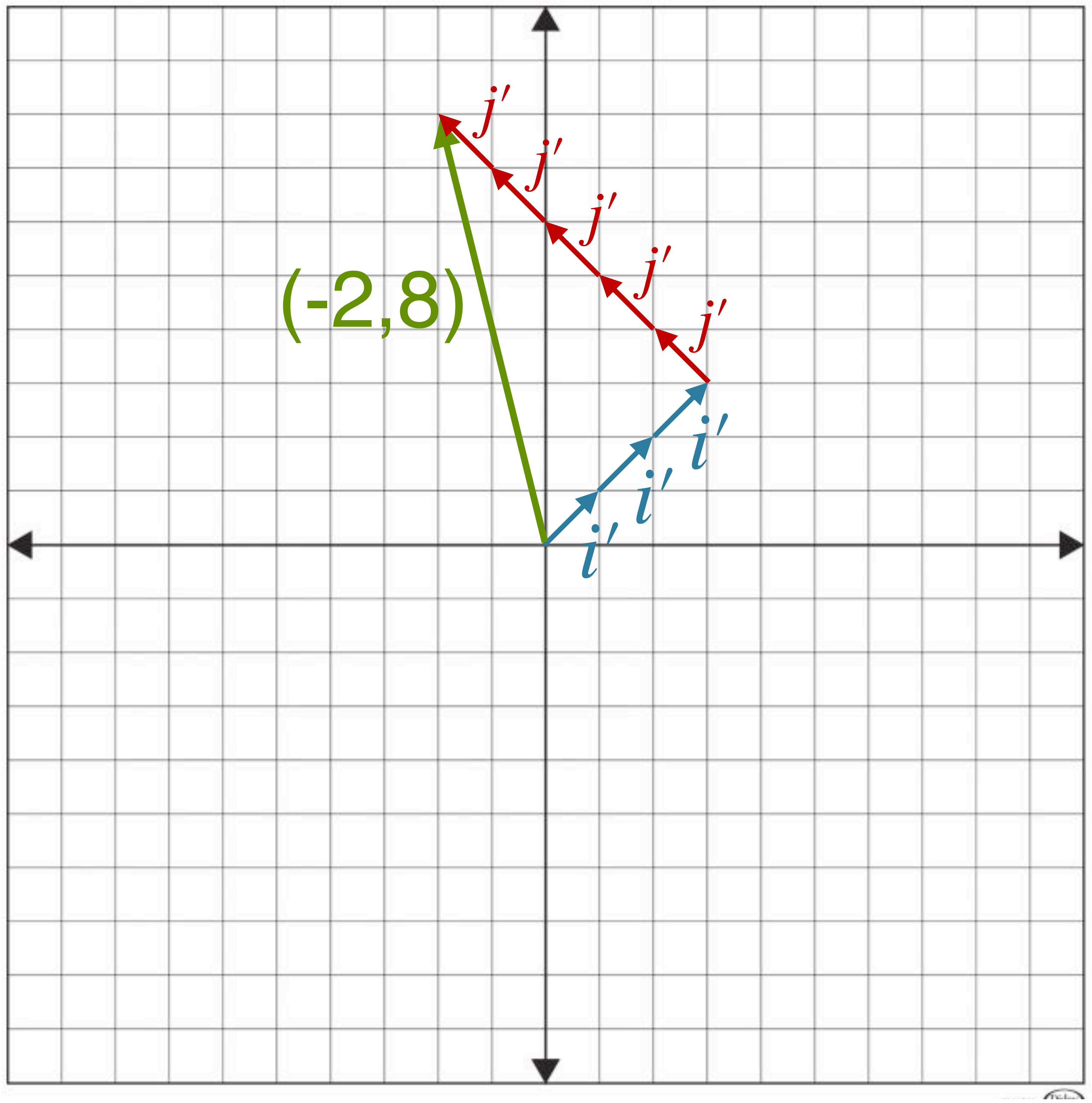

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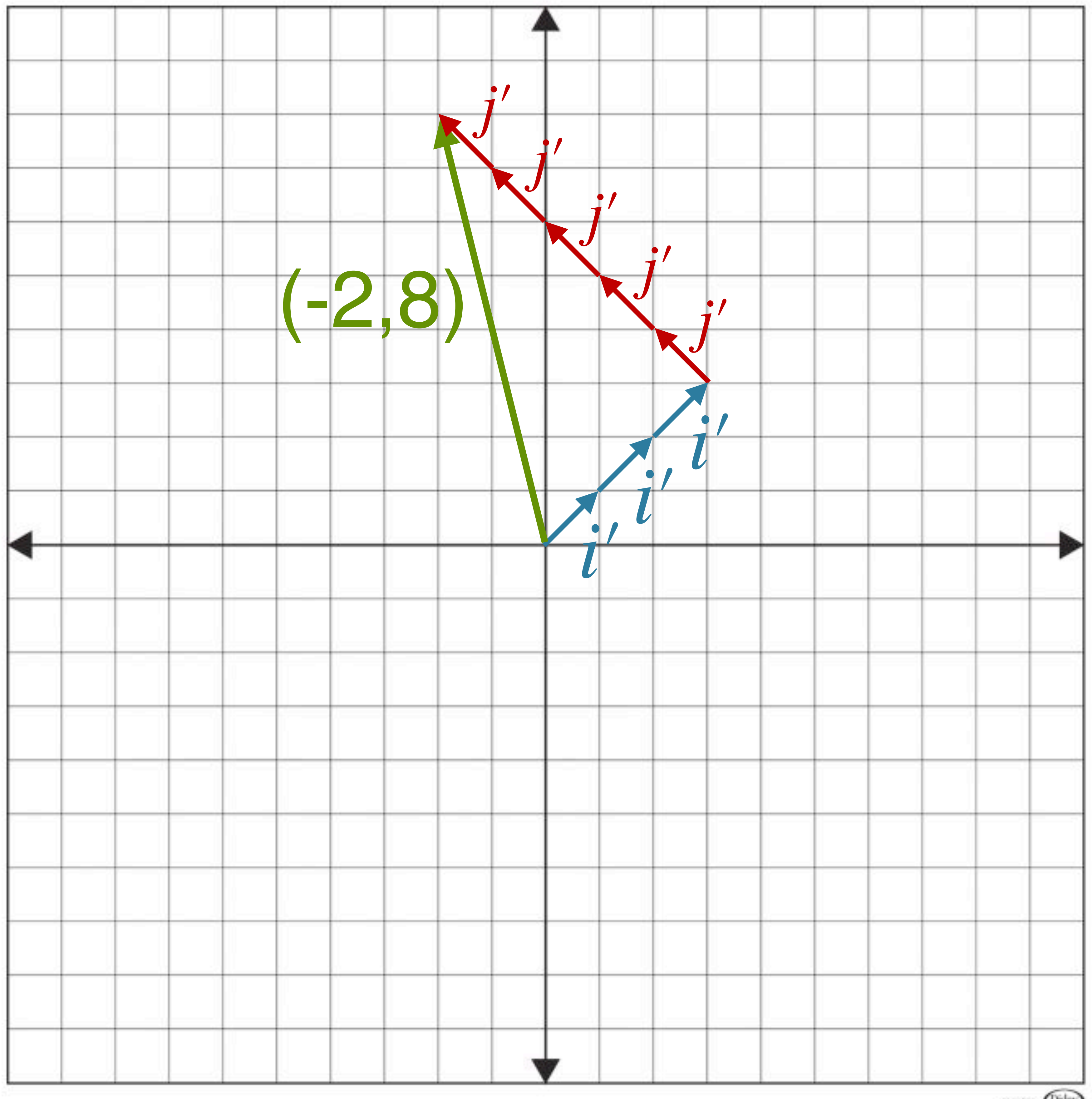

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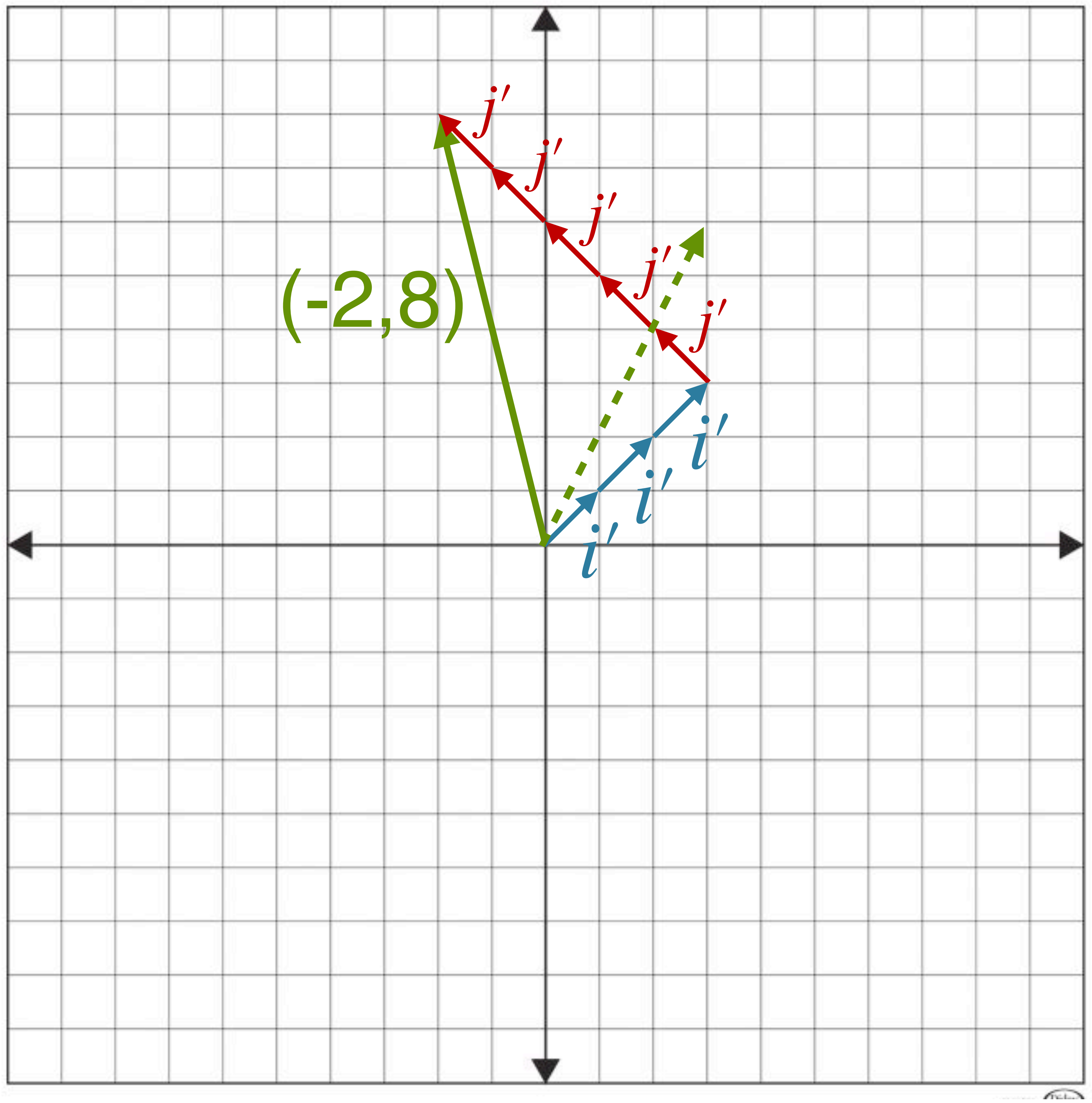

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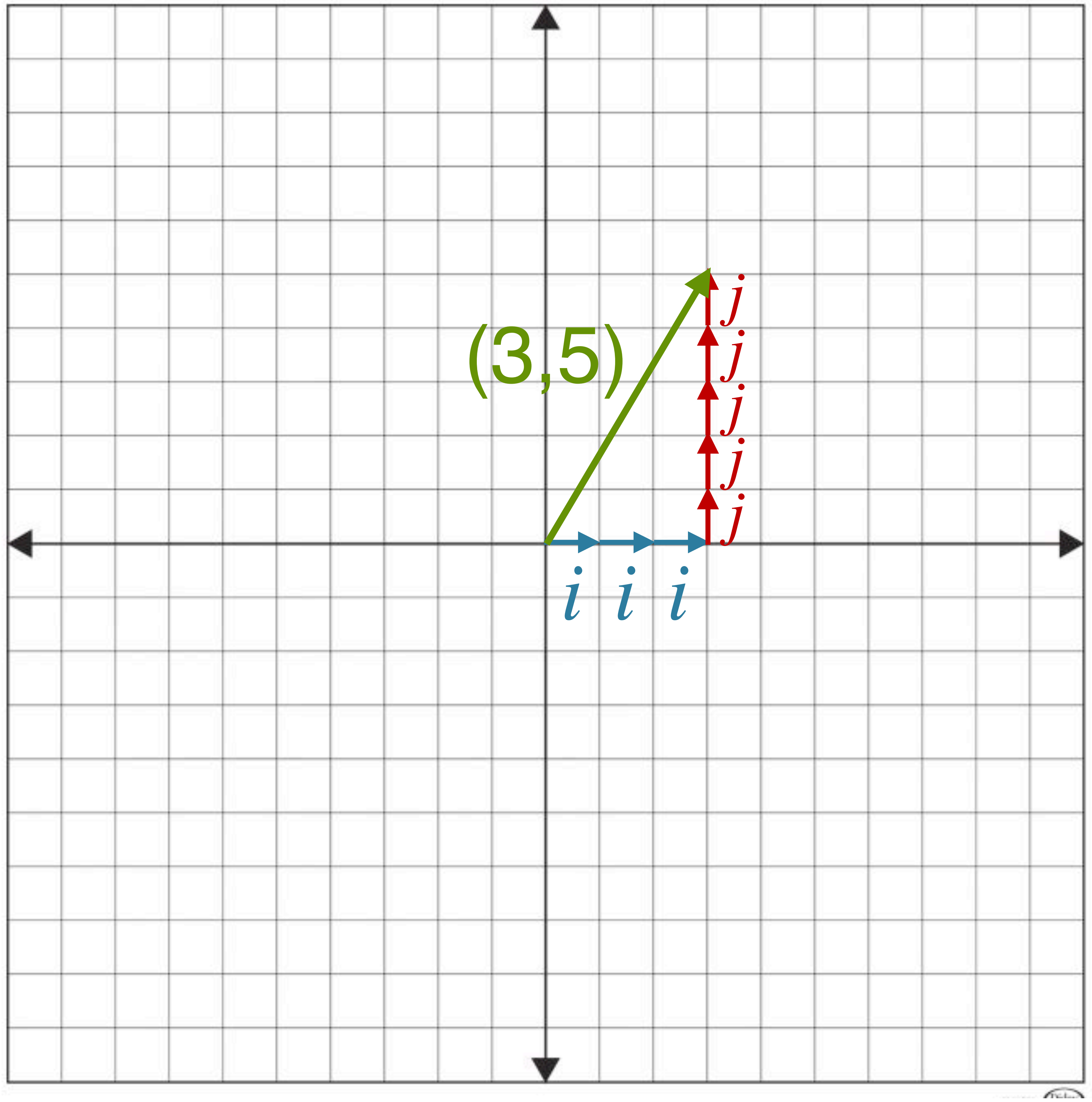
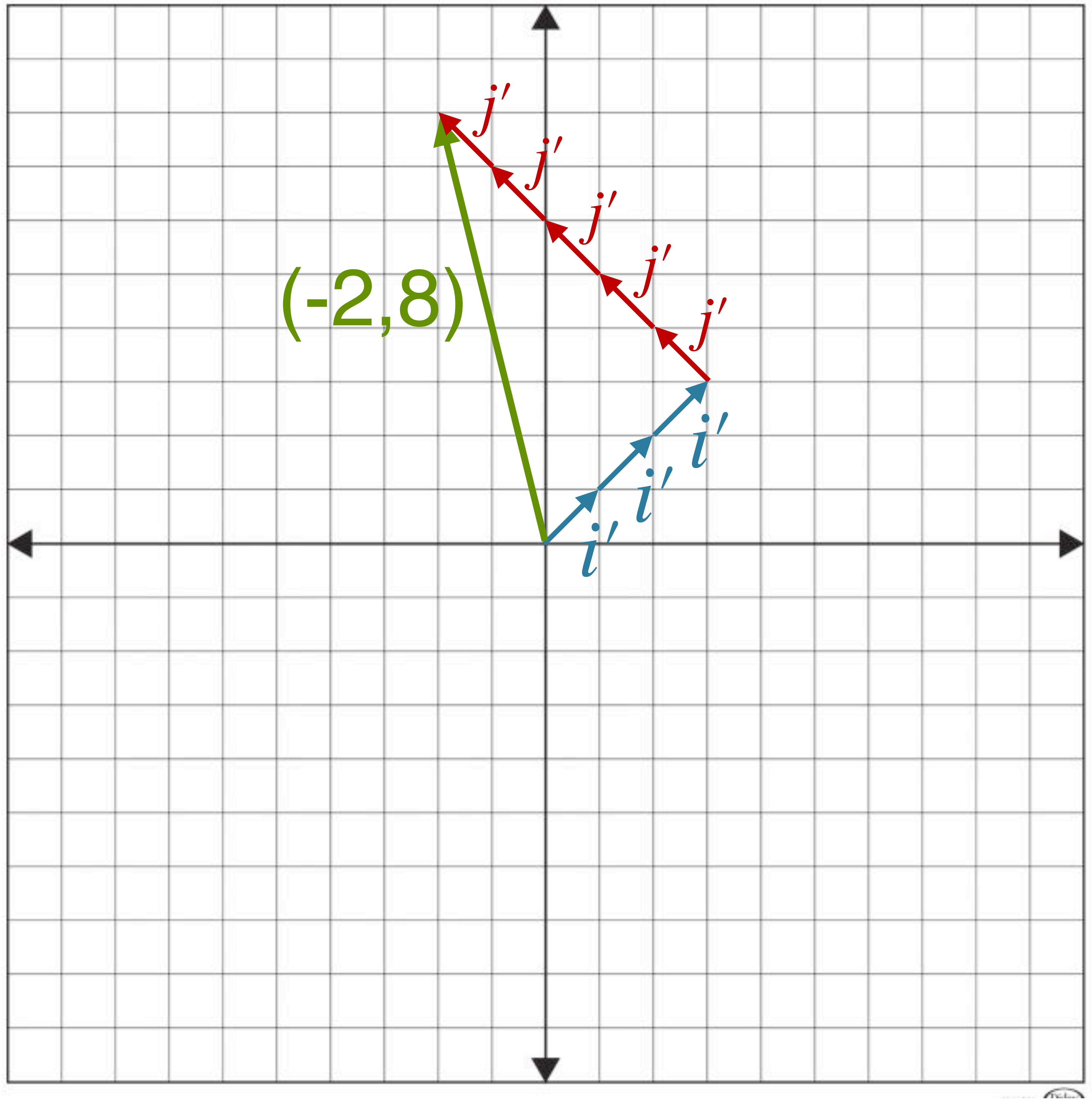
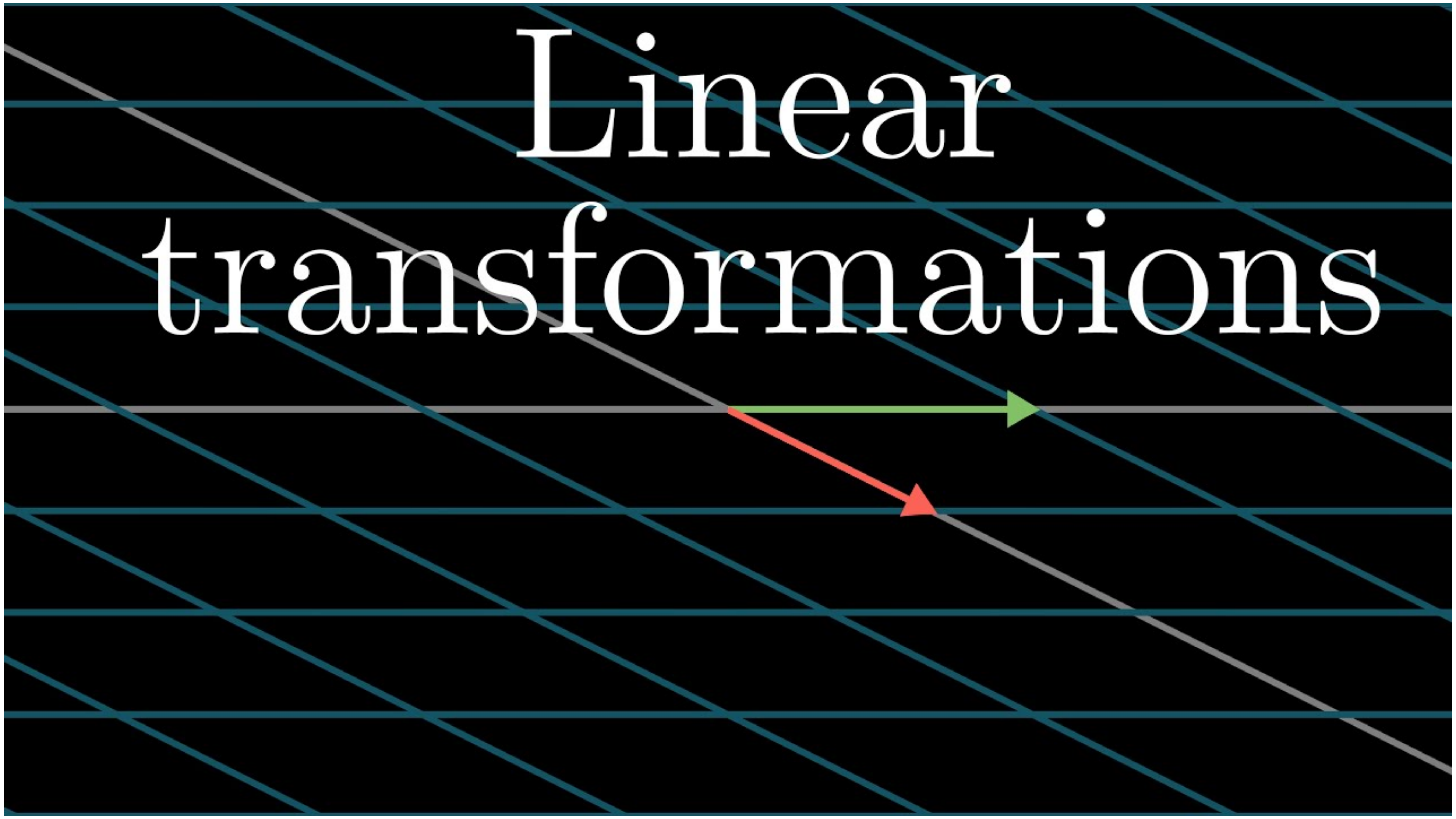




Visualizing Linear Transformations



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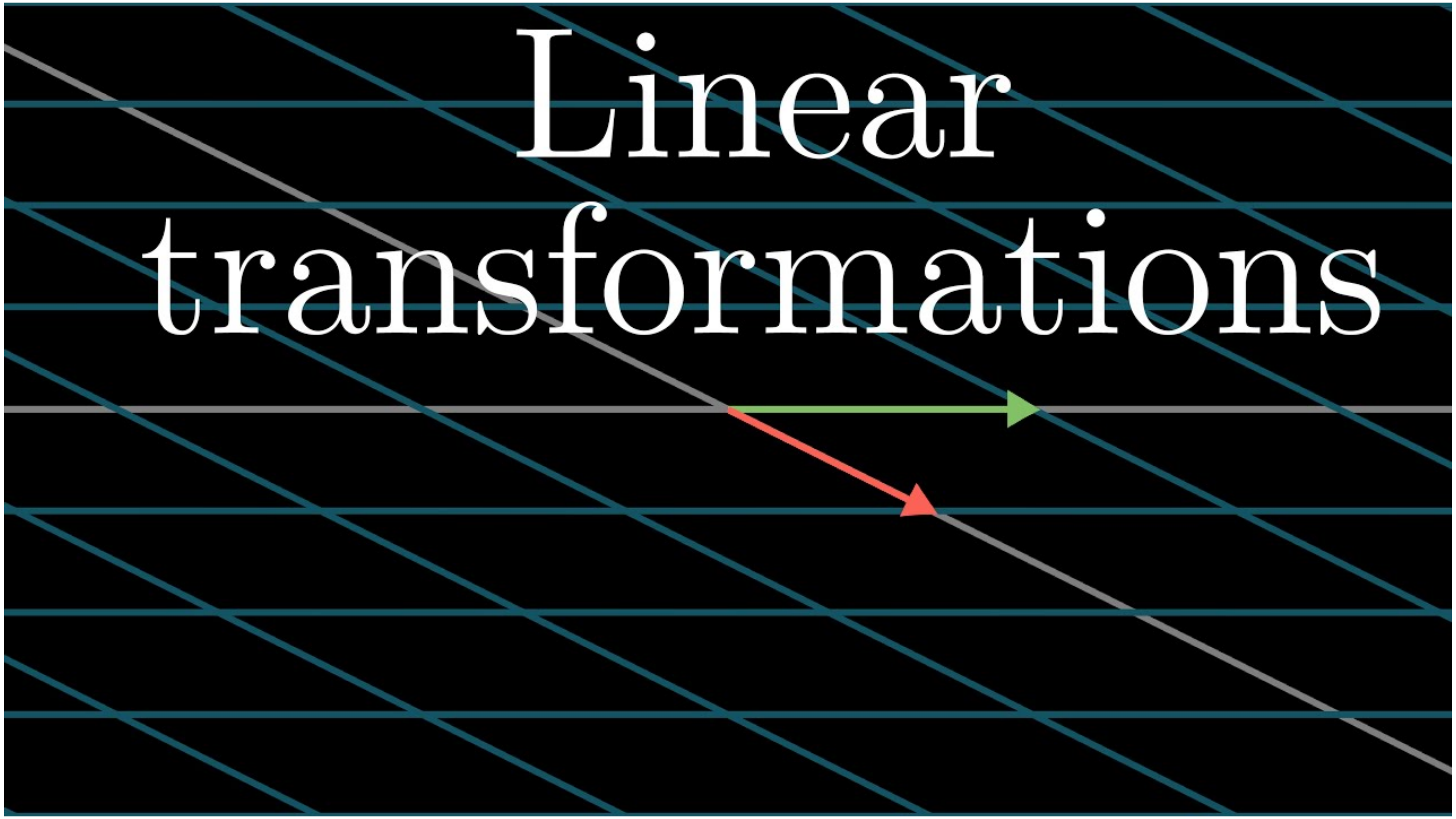

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- TLDR: Neural Nets transform vectors and vector spaces

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